A QUALITATIVE STUDY OF CONTROLLABILITY OF A CERTAIN CLASS OF FUZZY SYSTEMS AND NONLINEAR MATRIX LYAPUNOV SYSTEMS

A thesis submitted
in partial fulfillment for the degree of

Doctor of Philosophy

by

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CERTIFICATE

This is to certify that the thesis titled **'A Qualitative Study of Controllability of a Certain Class of Fuzzy Systems and Nonlinear Matrix Lyapunov Systems'**, submitted by **Bhaskar Dubey**, to the Indian Institute of Space Science and Technology, Thiruvananthapuram, for the award of the degree of **Doctor of Philosophy**, is a bonafide record of the research work done by him under my supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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DECLARATION

I declare that this thesis titled **'A Qualitative Study of Controllability of a Certain Class of Fuzzy Systems and Nonlinear Matrix Lyapunov Systems'** submitted in fulfillment of the degree of Doctor of Philosophy is a record of original work carried out by me under the supervision of **Prof. Raju K. George**, and has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this or any other Institution or University of higher learning. In keeping with the ethical practice in reporting scientific information, due acknowledgments have been made wherever the findings of others have been cited.

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Bhaskar Dubey

ABSTRACT

In the thesis, we study controllability of linear systems with fuzzy initial conditions and fuzzy inputs. Before establishing the controllability results for the fuzzy dynamical systems, we will first investigate the behavior of solutions of a general nonlinear system of ordinary differential equations with fuzzy initial conditions and fuzzy inputs. Although various approaches are suggested in the literature for the evolution of solution to fuzzy differential equations, we investigate controllability results by using the levelwise approach and differential inclusion approach. We also investigate controllability of nonlinearly perturbed matrix Lyapunov systems and impulsive semilinear matrix Lyapunov systems by using the tools of operator theory and nonlinear functional analysis.

The research work is mainly divided in to three parts. In the first part, we investigate the behavior of the solutions of fuzzy differential equations obtained by fuzzification of nonlinear ODEs with fuzzy initial conditions and fuzzy inputs. We consider the following n-dimensional nonlinear ordinary differential equations with fuzzy initial conditions and fuzzy inputs of the form

$$\dot{x}(t) = f(t, x(t), u(t)), x(t_0) = X_0, t \geq t_0 \geq 0, \tag{1}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a nonlinear function which is measurable in t and is continuous in x and u, $X_0 \in (\mathbb{E}^1)^n$ and the fuzzy input $u(t) \in (\mathbb{E}^1)^m$. In our analysis, we employ the tools of levelwise approach of solving fuzzy differential equations along with some of the results from real analysis. We have shown that the solutions of systems of type (1) are described by a system of $2n$-ordinary differential equations with crisp initial conditions and crisp inputs corresponding to the end points of the alpha cuts of fuzzy states.

We also consider a particular case of the systems of type (1), that is, linear time-varying dynamical systems with fuzzy initial conditions and fuzzy inputs of the

form

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ x(t_0) = X_0, t \geq t_0 \geq 0, \end{cases} \tag{2}$$

where $A(\cdot) \in C([t_0, t_1]; \mathbb{R}^{n\times n})$, $B(\cdot) \in C([t_0, t_1]; \mathbb{R}^{n\times m})$, $X_0 \in (\mathbb{E}^1)^n = \underbrace{\mathbb{E}^1 \times \ldots \times \mathbb{E}^1}_{n-times}$ and the input $u(t) \in (\mathbb{E}^1)^m$ for each $t \in [t_0, t_1](t_1 > t_0)$. Here we use a complex number representation of the α-level sets of the fuzzy states to characterize the solutions of such systems by a closed form formula involving the transition matrix which could be easily used in practical computations. We will use Peano-Baker type of series to obtain the transition matrix for the system (2).

We, further, consider fuzzy initial value problem of the type

$$\dot{x}(t) = f(t, x(t)), x(t_0) = x_0 \in (\mathbb{E}^1)^n, t_1 \geq t \geq t_0, \tag{3}$$

where $f : T = [t_0, t_1] \times (\mathbb{E}^1)^n \to (\mathbb{E}^1)^n$ is continuous, $t_0 \in \mathbb{R}^+$. We study the existence and uniqueness of the solution of system (3).

The second part of the thesis deals with the problems on controllability of linear fuzzy differential dynamical systems. Here, we first consider the following linear time invariant systems with fuzzy initial condition

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(t_0) = X_0, t_1 \geq t \geq t_0, \end{cases} \tag{4}$$

where A, B are real matrices of size $n \times n$, $n \times m$, respectively and $t_0 \in \mathbb{R}^+$. The initial state $X_0 \in (\mathbb{E}^1)^n$ and the control $u(t) \in (\mathbb{E}^1)^m$.

We study controllability of the system (4) by using the levelwise approach of evolution of solutions to system (4). In controllability, one looks for a fuzzy control $u(t)$ during time-interval $[t_0, t_1]$ such that the system can be steered exactly to a desired target fuzzy state X_1 at time t_1. That is, the solution of system (4) with the appropriate fuzzy control $u(t)$ during time interval $[t_0, t_1]$ satisfies $x(t_1) = X_1$, where X_1 is the desired fuzzy state at time t_1. We establish some sufficient conditions for the controllability of the system (4). We also provide a closed form representation for the steering control when the matrices A and B have non-negative entries. Furthermore, we introduce the concept of 'fuzzy-controllability', a concept weaker than controllability, and establish sufficient conditions for the fuzzy dynamical systems of type (4) to be fuzzy-controllable. In fuzzy-controllability, one looks for a fuzzy-controller $u(\cdot)$ that can steer the system-state within the desired target state X_1 at time t_1. More precisely, solution of system (4) with the fuzzy control $u(t)$ during time-interval $[t_0, t_1]$ satisfies $x(t_1) \leq X_1$, where X_1 is the desired fuzzy state at time t_1. In our work, we provide a computational procedure to obtain the fuzzy-controllable initial states that can be steered to within a desired target fuzzy state X_1 with some suitable control.

So far in our controllability analysis, we have employed the levelwise approach of the evolution of solutions of fuzzy differential equations in order to establish controllability results. We will now establish controllability results by using the differential inclusion approach. We consider the following time-varying systems of the form

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)U(t) \\ x(0) &= x_0 \in \mathbb{R}^n, T \geq t \geq 0, \end{aligned} \tag{5}$$

in which $A(t)$, $B(t)$ are $n \times n$, $n \times m$ continuous matrices, respectively. We assume that the control $u(t) \in (\mathbb{E}^1)^m$ and the state $x(t)$ for $t > 0$ belong to $(\mathbb{E}^1)^n$. Controllability of similar systems has been studied by other authors with an assumption of the invertibility of the matrix $B(t)$; while we obtain controllability results with a general non-invertible matrix $B(t)$. It is observed in our analysis that the system (5) may not be controllable on the whole space $(\mathbb{E}^1)^n$, instead controllability is established on a subset $\mathbb{E}_0^n$ of $(\mathbb{E}^1)^n$. This motivates us to introduce a concept of quasi-controllability, a weaker concept than controllability. We characterize the quasi-controllable subset $\mathbb{E}_0^n$ of $(\mathbb{E}^1)^n$ and establish sufficient conditions for quasi-controllability of system (5).

The third part of the thesis deals with the controllability analysis of the semilinear matrix Lyapunov systems and semilinear impulsive matrix Lyapunov systems. We use techniques from operator theory and nonlinear functional analysis to establish the complete controllability results for such systems. We study the controllability of nonlinear matrix Lyapunov systems represented by:

$$\dot{X}(t) = A(t)X(t) + X(t)B(t) + F(t)U(t) + G(t, X(t)), \tag{6}$$

where $X(t)$ is an $n \times n$ real matrix called state matrix, $U(t)$ is an $m \times n$ real matrix called control matrix and $G(\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is a nonlinear function. $A(t)$, $B(t)$, $F(t)$ are $n \times n$, $n \times n$ and $n \times m$ real matrices, respectively. All of them are assumed to be piecewise continuous functions of t in $[t_0, t_1](0 \leq t_0 < t_1 < \infty)$. Furthermore, entries in the state matrix $X(t)$ and the control matrix $U(t)$ belong to $L^2([t_0, t_1], \mathbb{R})$. The function G satisfies the 'Caratheodory conditions'; that is, $G(\cdot, x)$ is measurable with respect to t for all $x \in \mathbb{R}^{n\times n}$ and $G(t, \cdot)$ is continuous with respect to x for almost all $t \in [t_0, t_1]$. We establish our results under the assumption that nonlinear term $G(t, X(t))$ satisfies Lipschitz condition or monotonicity condition.

We also obtain sufficient conditions for the complete controllability of the following matrix Lyapunov systems with impulse effects

$$\begin{cases} \dot{X}(t) = A(t)X(t) + X(t)B(t) + F(t)U(t) + G(t, X(t)), t \neq t_k, t \in [t_0, T] \\ X(t_k^+) = [I_n + D^k U(t_k)]X(t_k), k = 1, 2, \dots, \rho \\ X(t_0) = X_0, \end{cases} \tag{7}$$

where the state $X(t)$ is an $n \times n$ real matrix, control $U(t)$ is an $m \times n$ real matrix. $A(t), B(t), F(t)$ are $n \times n$, $n \times n$, $n \times m$ real matrices with piecewise continuous entries in time interval $[t_0, T]$ and $0 \leq t_0 \leq t_1 \leq t_2 \ldots \leq t_\rho \leq T$ are the time points at which impulse control $U(t_k)$ is given to the system. For each $k = 1, 2, \dots, \rho$, $D^k U(t_k)$ is an $n \times n$ diagonal matrix such that $D^k U(t_k) = \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij}^k U_{ij}(t_k) I_n$, where I_n is the identity matrix on $\mathbb{R}^n$ and $d_{ij}^k \in \mathbb{R}$. $G(\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is a nonlinear function and satisfies the 'Caratheodory conditions'.

Controllability of special cases of system (6) and system (7) has been studied by several authors in the literature. However, our results are more general, applicable to a much wider class of systems and extend some of the existing results on the controllability of matrix Lyapunov systems and impulsive dynamical systems.

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ABBREVIATIONS

ODE	Ordinary Differential Equation
FDE	Fuzzy Differential Equation
MLS	Matrix Lyapunov Systems
IMLS	Impulsive Matrix Lyapunov Systems

NOTATION

$\mathbb{N}$	Natural numbers
$\mathbb{R}$	Real numbers
$\mathbb{R}^+$	Non-negative real numbers
$\mathbb{R}^n_+$	n-dimensional vectors of non-negative real numbers
$\mathbb{C}$	Complex numbers
$\mathbb{F}(X)$	Set of all fuzzy sets defined on the universe X
$\mathbb{E}$ or $\mathbb{E}^1$	Set of all fuzzy numbers defined on $\mathbb{R}$
$\mathbb{R}^{m\times n}$	Set of all $m \times n$ real matrices
$\mathbb{C}^{m\times n}$	Set of all $m \times n$ complex matrices
$\mathbb{E}^n$	Fuzzy numbers defined on $\mathbb{R}^n$
$(\mathbb{E}^1)^n$	n-dimensional vectors of fuzzy numbers defined on $\mathbb{R}$
$\mathbb{P}_k(X)$	Set of all compact and convex subsets of a reflexive Banach space X
$C([t_0, t_1]; X)$	Space of Banach space-X valued continuous functions on $[t_0, t_1]$

$L^2([t_0, t_1]; X)$	Space of Banach space-X valued square integrable functions on $[t_0, t_1]$
A^T	Transpose of the matrix A
$\|A\|$	2-norm of the matrix A
$\|A\|_F$	Frobenius norm of the matrix A
I_n	$n \times n$ identity matrix

CHAPTER 1

Introduction

1.1 General Introduction

Study of many of the real life systems in science and engineering domains are control theoretic in nature. In linear and nonlinear system theory, the problems related to controllability are always of the fundamental interest. Controllability analysis of such systems are essential in designing system parameters, computation of steering control, optimal control, etc (cf. Zabczyk (2008)). Though the controllability of the linear systems has been well investigated in the literature, controllability of the nonlinear systems is still one of the thriving and challenging area in the applied mathematics. Many authors have investigated the controllability of nonlinear systems, for example, George (1995), Hirschorn (1975), Joshi and George (1989) and many other authors. Furthermore, controllability of systems with uncertain or fuzzy parameters is relatively newer in the literature, for example, Ding and Kandel (2000a,b), Feng and Hu (2006) and Kwun et al. (2008) etc. Usually fuzziness or uncertainty may exist in the system due to the lack of precise measurements of parameters or sometimes due to the nature of the system.

Fuzzy sets and fuzzy systems theory have got tremendous applications in many of the real world problems including modelling and control of physical systems, designing knowledge based systems, intelligent systems etc. Starting right from the seminal work on fuzzy sets by Zadeh (1965), there is abundant literature on theoretical investigations on fuzzy systems theory and its practical applications. Specially the area of fuzzy differential equations has been enormously grown starting from its formal induction in to literature by Kandel and Byatt (1980). It is worthwhile to note that various approaches are proposed in the literature to define the solutions of fuzzy differential equations depending on the notion of the derivative involved in the equation, namely, Hukuhara or H-differentiability approach (cf. Puri and Ralescu (1983)), levelwise approach (cf.

Seikkala (1987)), differential inclusion approach (cf. Hüllermeier (1997), Ding and Kandel (2000a)), extension principle approach Buckley and Feuring (2000) and many others. Recently, Bede and Stefanini (2013) introduced the notion of generalized differentiability of fuzzy-valued functions (g-differentiability) by using the generalization of Hukuhara differences (gH-difference).

In our work, we mainly study controllability of the linear dynamical systems with fuzzy initial conditions and fuzzy inputs from the aspect of levelwise approach due to Seikkala (1987) and differential inclusion approach as in Ding and Kandel (2000a). We also investigate the behavior of solutions of fuzzy differential equations as they are essential for establishing the controllability results for fuzzy dynamical systems.

In our work, we will also investigate the complete controllability of nonlinear matrix Lyapunov systems and impulsive matrix Lyapunov systems. Matrix Lyapunov systems are regarded as the generalizations to usual dynamical systems and find applications in many engineering domains. Recently, Murty et al. (2006) have studied the complete controllability of linear matrix Lyapunov systems, also some stability property for the linear matrix Lyapunov systems is investigated in Murty and Kumar (2008b). Their results motivate the study of controllability of nonlinear matrix Lyapunov systems.

Furthermore, we will establish complete controllability of impulsive matrix Lyapunov systems both linear and nonlinear. Impulsive systems have been proved very effective in modelling of the physical systems that are subject to sudden changes in control or state at discrete time points. Many evolutionary processes, for instance, some motions of aircrafts and satellites, system models in economics, frequency modulated systems and bursting rhythm models in biology are impulsive in nature. We establish various sufficient conditions for the controllability of impulsive matrix Lyapunov systems both linear and semilinear.

Thus, the main objectives of the thesis is to investigate the following problems:

Problem 1: Investigation of the evolution of solution to system of linear and nonlinear ordinary differential equations with fuzzy initial conditions and fuzzy inputs.

Problem 2: Investigation of the controllability property of linear systems with fuzzy initial conditions and fuzzy inputs. We establish our results by using both levelwise approach and differential inclusion approach.

Problem 3: Controllability study of semilinear matrix Lyapunov systems .

Problem 4: Controllability study of impulsive matrix Lyapunov systems, both linear and semilinear.

We shall now briefly describe each of the problems. We consider an n-dimensional nonlinear system of ordinary differential equations with fuzzy initial conditions and fuzzy inputs

$$\dot{x}(t) = f(t, x(t), u(t)), x(t_0) = X_0, t \geq t_0 \geq 0, \tag{1.1}$$

where $f : R^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a nonlinear function which is measurable in t and is continuous in x and u. The fuzzy initial condition X_0 is an n-vector of fuzzy numbers on $\mathbb{R}$, and the fuzzy input $u(t)$ is an m-vector of fuzzy numbers on $\mathbb{R}$. By using the Zadeh's extension principle, we prove that the evolution of system (1.1) is described by a system of $2n$-ordinary differential equations corresponding to the end points of the α-cuts of x.

We also consider a special case of the system (1.1) of the type

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ x(t_0) = X_0, t_1 \geq t \geq t_0, \end{cases} \tag{1.2}$$

where $A(\cdot) \in C([t_0, t_1]; \mathbb{R}^{n\times n}), B(\cdot) \in C([t_0, t_1]; \mathbb{R}^{n\times m}), X_0 \in (\mathbb{E}^1)^n = \underbrace{\mathbb{E}^1 \times \ldots \times \mathbb{E}^1}_{n-times}$, $t_0 \in \mathbb{R}^+$, the input $u(t) \in (\mathbb{E}^1)^m$ for each $t \in [t_0, t_1](t_1 > t_0)$, moreover $u(\cdot)$ is assumed to be integrable in $[t_0, t_1]$. We discuss the existence and uniqueness of system (1.2). We use a complex number representation of the α-level sets of the fuzzy states and a generalization to Peano-Baker series to obtain the transition matrix in order to characterize the solutions of such systems by a closed form formula which could be easily used in practical computations. Xu et al. (2007) have also used such closed form representation of solutions by using complex number representation of α-cuts of fuzzy states for time invariant systems of type (1.2), thats is, $A(t) = A$ and $B(t) = B$ in Eq. (1.2).

In the investigation of controllability property of fuzzy dynamical systems, we mainly focus on the time invariant systems of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(t_0) = X_0, t \geq t_0 \geq 0, \end{cases} \tag{1.3}$$

where A, B are real matrices of size $n \times n$, $n \times m$, respectively. The initial state $X_0 \in (\mathbb{E}^1)^n$ (hence $x(t) \in (\mathbb{E}^1)^n$)and the control $u(t) \in (\mathbb{E}^1)^m$. Although the analysis of controllability of various fuzzy systems, namely, T-S fuzzy system, (cf. Biglarbegian et al. (2012), Chen et al. (2009) and references their in), fuzzy logic systems (cf. Gupta et al. (1986), Farinwata and Vachtsevanos (1993)) has been long standing in the literature, there is relatively a modest attempt towards the analysis of controllability of fuzzy dynamical system from the aspect of fuzzy differential equation. For our purpose, we refer to Ding and Kandel (2000a,b), Feng and Hu (2006), Murty and Kumar (2008a), Kwun et al. (2009, 2008). In our work, we establish sufficient conditions for the controllability of system (1.3). In controllability, one looks for a fuzzy control input $u(t) \in (\mathbb{E}^1)^m$ such that given a desired target fuzzy state X_1 at time t_1 solution of system (1.3) satisfies $x(t_1) = X_1$. In our analysis, we found that the property of controllability for the fuzzy systems of type (1.3) is stronger than the controllability property for the crisp systems, that is, system (1.3) with crisp initial condition. It has also been observed that not all state in $(\mathbb{E}^1)^n$ are reachable from the given initial state $X_0 \in (\mathbb{E}^1)^n$ even if the pair (A, B) is controllable. Thus, we provide a characterization of reachable states from the given initial state $X_0 \in (\mathbb{E}^1)^n$ and computation of steering control that steers an initial fuzzy state $X_0 \in (\mathbb{E}^1)^n$ to a desired target state $X_1 \in (\mathbb{E}^1)^n$.

We also introduce the concept of fuzzy-controllability, a concept weaker than controllability, and establish sufficient conditions for the system (1.3) to be fuzzy-controllable. The problem which we deal with fuzzy-controllability is as follows. Let x_0 and x_1 be two crisp vectors in $\mathbb{R}^n$ and $X_1 \in (\mathbb{E}^1)^n$ be the fuzzy state at time t_1 around x_1, then find out an initial fuzzy state X_0 at t_0 around x_0 and some suitable fuzzy control inputs such that the solution of system (1.3) at t_1 satisfies $x(t_1) \leq X_1$. Note that we do not require exact equality between propagated system state at time t_1 and desired target state X_1, instead we want the the system state at time t_1 to be restricted within X_1. Roughly speaking, as also desired in many practical systems, we do not want the system state to be controlled to a precise point in the state space so long as minimal requirement is satisfied. For example, in the game of basketball, where the player's primary interest is to throw the ball in the basket from the base line. Player is not quite interested whether the center of the ball coincides with that of basket while passing through it. Here the position of ball from center of the basket form the target fuzzy set, thus the problem is to find initial membership grade around the baseline and some fuzzy control input so that the ball can be placed in the basket. Similar situations can occur in other applications like, air conditioning system, biological systems, satellite injection in an orbit and orbit manoeuvering of satellites, etc.

We will also investigate controllability of the following time-varying fuzzy dynamic control systems of the form

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ x(0) = x_0 \in \mathbb{R}^n, T \geq t \geq 0, \end{cases} \tag{1.4}$$

where $A(\cdot) \in C([0,T];\mathbb{R}^{n\times n})$, $B(\cdot) \in C([0,T];\mathbb{R}^{n\times m})$, and the control $u(t) \in (\mathbb{E}^1)^m$.

Ding and Kandel (2000a) investigated the controllability of system (1.4) with $B(\cdot) \in C([0,T];\mathbb{R}^{n\times n})$. Moreover they have assumed the invertibility of the matrix $B(t)$, which is quite strong assumption and can not be often met in practical systems. Therefore, we establish our results on controllability for a general matrix $B(t)$ of size $n \times m$. In our analysis, we have observed when $B(t)$ is of size $n \times m$ complete controllability can not be guaranteed, that is, it may not be possible to control any arbitrary state $X \in (\mathbb{E}^1)^n$ from the initial state x_0. Thus, we characterize $\mathbb{E}_0^n$, a subset of $(\mathbb{E}^1)^n$, called the set of all admissible controllable states in $(\mathbb{E}^1)^n$.

Now we shall describe our problems dealing with the controllability of nonlinear matrix Lyapunov systems and impulsive matrix Lyapunov systems. First, we investigate the controllability of nonlinear matrix Lyapunov systems represented by:

$$\dot{X}(t) = A(t)X(t) + X(t)B(t) + F(t)U(t) + G(t, X(t)), \tag{1.5}$$

where $X(t)$ is an $n \times n$ real matrix called state matrix, $U(t)$ is an $m \times n$ real matrix called control matrix and $G(\cdot, \cdot) : \mathbb{R}^+ \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is a nonlinear function. $A(t)$, $B(t)$, $F(t)$ are $n \times n$, $n \times n$ and $n \times m$ real matrices respectively. All of them are assumed to be piecewise continuous function of t in $[t_0, t_1](t_0 < t_1 < \infty)$. Furthermore, entries in the state matrix $X(t)$ and the control matrix $U(t)$ belong to $L^2([t_0, t_1], \mathbb{R})$. The function G satisfies the 'Caratheodory conditions'; that is, $G(\cdot, x)$ is measurable with respect to t for all $x \in \mathbb{R}^{n\times n}$ and $G(t, \cdot)$ is continuous with respect to x for almost all $t \in [t_0, t_1]$.

Recently, Murty et al. (2006) have studied the controllability of linear matrix Lyapunov systems, that is, the system (1.5) with $G(t, x) = 0$. Furthermore, if $G(t, x) \equiv 0$ and $B(t) \equiv 0$ then the system (1.5) reduces to linear time-varying control system whose controllability is well established in the literature, for example, Barnett and Cameron (1975), Sontag (1998). We establish many sufficient conditions for the complete controllability of system (1.5) by using the tools of operator theory, nonlinear functional analysis and fixed point theorems. In our analysis we assume that nonlinear function G either satisfies Lipschitz condition or monotonicity condition.

We will also investigate complete controllability of the following matrix Lyapunov systems with impulse effects

$$\begin{cases} \dot{X}(t) = A(t)X(t) + X(t)B(t) + F(t)U(t) + G(t, X(t)), t \neq t_k, t \in [t_0, T] \\ X(t_k^+) = [I_n + D^k U(t_k)]X(t_k), k = 1, 2, \ldots, \rho \\ X(t_0) = X_0, \end{cases} \tag{1.6}$$

where the state $X(t)$ is an $n \times n$ real matrix, control $U(t)$ is an $m \times n$ real matrix. $A(t), B(t), F(t)$ are $n \times n$, $n \times n$, $n \times m$ real matrices with piecewise continuous entries and $t_0 \leq t_1 \leq t_2 \ldots \leq t_\rho \leq T$ are the time points at which impulse control $U(t_k)$ is given to the system. For each $k = 1, 2, \ldots, \rho$, $D^k U(t_k)$ is an $n \times n$ diagonal matrix such that $D^k U(t_k) = \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij}^k U_{ij}(t_k) I_n$, where I_n is the identity matrix on $\mathbb{R}^n$ and $d_{ij}^k \in \mathbb{R}$. $G(\cdot, \cdot) : \mathbb{R}^+ \times \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ is a nonlinear function and satisfies the 'Caratheodory conditions'. The control $U(t)$ is said to be impulsive if at $t = t_k, k = 1, 2, \ldots, \rho$, the pulses are regulated and chosen arbitrarily in rest of the domain.

Indeed, the controllability of many special cases of system (1.6) has been studied in the literature. For example, if $B(t) = 0$ and $G(t, x) = 0$ hold along with $D^k U(t_k) = 0$ for $k = 1, 2, \ldots, \rho$, then the system (1.6) reduces to linear time-varying control system whose controllability is well established in the literature, for example, Russell (1979), Zabczyk (2008). Leela et al. (1993) studied the controllability of a special case of (1.6), that is, when $B(t) = 0$, $G(t, x) = 0$, and $A(t)$, $F(t)$ are constant matrices. Also, George et al. (2000), established complete controllability of a special case of system (1.6) with $B(t) = 0$.

Murty et al. (2006) investigated the controllability of linear non-impulsive matrix Lypunov systems, that is, system (1.6) with $G = 0$ and without impulses. Furthermore, in Dubey and George (2013b) complete controllability of semilinear non-impulsive matrix Lyapunov systems is established.

In our work, first we obtain sufficient conditions for the complete controllability of linear IMLS, that is, system (1.6) with $G = 0$. We, then, establish complete controllability of semilinear IMLS, that is, system (1.6) itself. In our analysis, we assume that nonlinearities G are either Lipschitzian or monotone.

1.2 Thesis Outline

We provide necessary concepts of control systems theory, fuzzy sets and fuzzy system theory in Chapter 2.

Chapter 3 deals with the study of qualitative behavior of the solutions of a system of nonlinear ordinary differential equations with fuzzy initial conditions and fuzzy inputs by using the tools of levelwise apporach. Our results generalize some of the results in the literature, namely, Kaleva (1987), Seikkala (1987), Xu et al. (2007). As a particular case, for the linear systems with fuzzy initial conditions and fuzzy inputs, we provide a closed form representation for the solutions of such systems. We also study the differentiability and integrability properties of fuzzy vector-valued functions whose values are n-dimensional vectors of fuzzy numbers on $\mathbb{R}$. We establish sufficient conditions for the existence and uniqueness of a solution to a system of fuzzy differential equations in which derivative is given by such functions. We provide examples to illustrate our results.

In Chapter 4, we study controllability property for the linear systems with fuzzy initial conditions and fuzzy inputs. We establish sufficient conditions for the system to be controllable. We study controllability under both, the levelwise approach of evolution of solutions of underlying systems as in Seikkala (1987) and the differential inclusion approach as in Ding and Kandel (2000a). We obtain many new results on the investigation of controllability property via levelwise approach; our results generalize and extend some of the results in Feng and Hu (2006).We also introduce a concept of fuzzy-controllability, a concept weaker than controllability, for the linear fuzzy systems and provide a computational procedure for the estimation of fuzzy-controllable initial states. Few examples are provided to substantiate the results obtained.

Our results on the controllability via differential inclusion approach are the extension of the controllability results due to Ding and Kandel (2000a,b). In particular, we weakened the assumption of invertibility of the input-to-state matrix $B(t)$ assumed in Ding and Kandel (2000a).

Chapter 5 deals with the controllability analysis of nonlinear matrix Lyapunov systems. We discuss mainly the controllability of semilinear matrix Lyapunov systems with two types of nonlinearities - Lipschitzian and monotone. We establish sufficient conditions for the complete controllability of the nonlinear matrix Lyapunov systems under both Lipschitzian and monotone nonlinearities. An examples is also provided to illustrate the results obtained.

In Chapter 6, we investigate complete controllability of linear and nonlinear impulsive matrix Lyapunov systems. First, we establish the complete controllability of linear impulsive matrix Lyapunov systems. We, then, consider the semilinear impulsive matrix Lyapunov systems with both Lipschitzian and monotone nonlinearities. Many sufficient conditions are obtained for the semilinear impulsive matrix Lyapunov systems to be controllable. Examples are given to illustrate the results.

CHAPTER 2

Preliminaries

2.1 Basics of Control Theory

In our investigation, the controllability properties of various systems, namely, fuzzy dynamical systems, nonlinear matrix Lyapunov Systems and impulsive matrix Lyapunov systems depend more on the properties of the following linear system represented by the equation

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ x(t_0) = x_0 \in \mathbb{R}^n, t \geq t_0 \geq 0, \end{cases} \tag{2.1}$$

where $A(\cdot) \in C([t_0, t_1]; \mathbb{R}^{n\times n})$ and $B(\cdot) \in C([t_0, t_1]; \mathbb{R}^{n\times m})$, $t_1 > t_0$. The matrix $A(t)$ is sometimes called the plant matrix and the matrix $B(t)$ is called the control matrix. For each $t \in [t_0, t_1]$, $x(t) \in \mathbb{R}^n$ is called the state of the system and $u(\cdot) \in L^2([t_0, t_1]; \mathbb{R}^m)$ is called the input or the control function for the system. If the matrices $A(t)$ and $B(t)$ are matrices with constant entries then system (2.1) is called linear time-invariant dynamical system. The unique solution of the system (2.1) is given by the following equation(cf. Szidarovszky (1998)):

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau, \tag{2.2}$$

where $\Phi(t, \tau)$ is the transition matrix for the homogeneous system $\dot{x} = A(t)x$.

Definition 2.1.1. *(Transition Matrix) The transition matrix $\Phi(t, \tau)$ for the system* (2.1) *is regarded as the unique solution of following matrix differential equation:*

$$\dot{X}(t) = A(t)X(t), \quad X(\tau) = I_n,$$

where I_n is the $n \times n$ identity matrix.

The transition matrix provides a transition of the system states; in other words future states are directly computed using the transition matrix. That is, the unique solution of the system $\dot{x} = A(t)x, x(t_0) = x_0$, is given by

$$x(t) = \Phi(t, t_0)x_0. \tag{2.3}$$

$\Phi(t, t_0)$ is also expressed by the following Piano-Baker series:

$$\Phi(t, t_0) = I_n + \int_{t_0}^{t} A(\tau)\mathrm{d}\tau + \int_{t_0}^{t} A(\tau) \int_{t_0}^{\tau} A(s)\mathrm{d}s\mathrm{d}\tau + \\ \int_{t_0}^{t} A(\tau) \int_{t_0}^{\tau} A(s) \int_{t_0}^{s} A(w)\mathrm{d}w\mathrm{d}s\mathrm{d}\tau \ldots.$$

Transition matrix satisfies the following important properties (cf. Szidarovszky (1998)):

1. $\Phi(t, t) = I$.
2. $\Phi(t, \tau)\Phi(\tau, s) = \Phi(t, s)$.
3. $\Phi(t, \tau)^{-1} = \Phi(\tau, t)$.
4. $\frac{\partial}{\partial t}\Phi(t, \tau) = A(t)\Phi(t, \tau)$.
5. $\frac{\partial}{\partial \tau}\Phi(t, \tau) = -\Phi(t, \tau)A(\tau)$.

We will now define the controllability for the system (2.1). Although there are various equivalent definition for the controllability of the system (2.1), our definition is as follows (refer to Szidarovszky (1998), Russell (1979)).

Definition 2.1.2. *(Controllability) The system* (2.1) *is said to be controllable to a state* $x_1 \in \mathbb{R}^n$ *at time* $t_1 (> t_0)$ *if there exists a control* $u(\cdot) \in L^2([t_0, t_1]; \mathbb{R}^m)$ *such that the solution of system* (2.1) *also satisfies* $x(t_1) = X_1$. *That is,*

$$x_1 = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau.$$

Remark 2.1.3. *If the initial state* x_0 *at* t_0 *and desired target state* x_1 *at* t_1 *are chosen arbitrarily in* $\mathbb{R}^n$ *then the system is called* $\mathit{completely\ or\ globally\ controllable}$ *during* *time-interval* $[t_0, t_1]$. *If* x_0 *and* x_1 *are required only to belong to a subset* $D \subset \mathbb{R}^n$, *then the resulting controllability is called* $local\ controllability$.

Now we will provide characterization of complete controllability of system (2.1). Define an operator $\mathcal{C} : L^2([t_0, t_1]; \mathbb{R}^m) \to \mathbb{R}^n$ by

$$\mathcal{C}u = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau.$$

It is clear that the system (2.1) is completely controllable if and only if the operator $\mathcal{C}$ (called control operator) is surjective.

Let $\mathcal{C}^* : \mathbb{R}^n \to L^2([t_0, t_1], \mathbb{R}^m)$ denote the adjoint of $\mathcal{C}$ and is defined by

$$(\mathcal{C}^* x)(t) = B^T(t)\Phi^T(t_1, t)x, x \in \mathbb{R}^n.$$

It can be easily shown that the operator $\mathcal{C}$ is onto iff the operator $\mathcal{C}\mathcal{C}^*$ is onto. The operator $\mathcal{C}\mathcal{C}^*$ is also called controllability Grammian $W(t_0, t_1)$ and is defined by:

$$\mathcal{C}\mathcal{C}^* = W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)B^T(\tau)\Phi^T(t_1, \tau)d\tau.$$

The above discussion can be summarized in the form of following important theorem.

Theorem 2.1.4. *The following statements are equivalent*:

(i) *System* (2.1) *is completely controllable.*

(ii) $\mathcal{C}$ *is surjective.*

(iii) $\mathcal{C}^*$ *is injective.*

(iv) $\mathcal{C}\mathcal{C}^*$ *is surjective.*

If $W(t_0, t_1)$ is invertible, then it can be easily shown that the control given by $\mathcal{C}^*(\mathcal{C}\mathcal{C}^*)^{-1}(x_1 - \Phi(t_1, t_0)x_0)$ steers the system from initial state x_0 at time t_0 to a desired state x_1 at time t_1. More precisely the control $u_0(t)$, $t \geq t_0$, defined by

$$u_0(t) = B^T(t)\Phi^T(t_1, t)W^{-1}(t_0, t_1)(x_1 - \Phi(t_1, t_0)x_0) \tag{2.4}$$

steers the system state from x_0 to x_1 during time interval $[t_0, t_1]$.

Furthermore, it can be shown that the steering control $u_0(t)$ is an optimal control. The following theorem depicts this fact.

Theorem 2.1.5. *(Minimum norm control) The control function $u_0(\cdot)$ as defined in Eq.* (2.4) *has minimum L^2 norm (energy) among all possible controls $u \in L^2([t_0, t_1]; \mathbb{R}^m)$ steering the system state x_0 to x_1 during time interval $[t_0, t_1]$.*

Although invertibility of controllability Grammian is necessary and sufficient condition for the system (2.1) to be completely controllable, it is not computationally efficient. However, for time invariant systems of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(t_0) = x_0 \in \mathbb{R}^n, t \geq t_0 \geq 0, \end{cases} \tag{2.5}$$

we have the following simple characterization of controllability.

Theorem 2.1.6. *The following statements are equivalent.*

1. *The system* (2.5) *is completely controllable.*
2. *(Kalman rank condition)*

$$rank([B\,AB\,A^2B\,\ldots A^{n-1}B]) = n.$$

3. *No eigenvector of A^T lies in the kernal of B^T.*
4. *(PBH Test) rank[A-λI B] = n for every eigenvalue λ of A.*

2.2 Basics of Fuzzy Sets and Systems

The following definition of fuzzy sets is due to Zimmermann (2001).

Definition 2.2.1. *If X is a collection of objects denoted generically by x, then a fuzzy set A in X is a set of ordered pairs $A = \{(x, \mu_A(x)) | x \in X\}$, where $\mu_A : X \to [0, 1]$*

is called the membership function and $\mu_A(x)$ is called the grade of membership of x in A. For the sake of convenience, we use the notation $A(x)$ to denote $\mu_A(x)$.

Definition 2.2.2. *Let A be a fuzzy set defined on the universe X. For $\alpha \in (0, 1]$, the $\alpha-$cut or $\alpha-$level set of A, denoted by A^α or $[A]_\alpha$, is defined as $A^\alpha = \{x \in X : A(x) \geq \alpha\}$. For $\alpha = 0$, the 0-cut of A is defined as the closure of the union of all non-zero α-cuts of A. That is,*

$$A^0 = \overline{\bigcup_{\alpha \in (0,1]} A^\alpha}.$$

Definition 2.2.3. *(Zadeh (1965)) A fuzzy set A defined on $\mathbb{R}^n$ is convex if and only if the sets A^α as defined above are convex for all α in interval $(0, 1]$.*

An alternative and more direct definition of the convexity is the following:

Definition 2.2.4. *A fuzzy set A defined on $\mathbb{R}^n$ is convex if and only if*

$$A(\lambda x_1 + (1 - \lambda)x_2) \geq \min(A(x_1), A(x_2)),$$

for all $x_1, x_2 \in \mathbb{R}^n$ and $\lambda \in [0, 1]$. Note that it is assumed that $A(x) = 0$ for $x \in \mathbb{R}^n \setminus A^0$ in case if A is defined only on a proper subset of $\mathbb{R}^n$.

It is to be noted that the above definition does not imply that $A(x)$ must be a convex function of x.

Definition 2.2.5. *Let $\{A_i\}_{i\in\Lambda}$ be a family of fuzzy sets on X, where Λ is any arbitrary index set. Then the fuzzy union of $\{A_i\}$ is denoted by the fuzzy set $\bigcup_{i\in\Lambda} A_i$ and is defined as follows:*

$$\big(\bigcup_{i\in\Lambda} A_i\big)(x) = \sup\{A_i(x)\},\ x \in X.$$

Fuzzy intersection of $\{A_i\}$ is denoted by $\bigcap_{i\in\Lambda} A_i$ or $\bigwedge_{i\in\Lambda} A_i$ and defined as follows:

$$\big(\bigcap_{i\in\Lambda} A_i\big)(x) = \inf\{A_i(x)\},\ x \in X.$$

Every fuzzy set can be uniquely represented in terms of its $\alpha-$cuts. The following decomposition theorem of fuzzy sets supports this fact.

Theorem 2.2.6. *Let X be an arbitrary set. Then for every $A \in \mathbb{F}(X)$, the set of all fuzzy sets on X, we have $A = \bigcup_{\alpha\in[0,1]} {}_{\alpha}A$, in which $\bigcup$ denotes the standard fuzzy union and for $\alpha \in [0,1]$, the fuzzy sets ${}_{\alpha}A$ are defined by*:

$$
{}_{\alpha}A(x) = \begin{cases} \alpha & \text{if } x \in A^{\alpha} \\ 0 & \text{if } x \in X \setminus A^{\alpha}. \end{cases} \tag{2.6}
$$

(For proof, refer to Klir and Yuan (1995))

We shall now define fuzzy numbers, a special type of fuzzy sets on $\mathbb{R}$.

Definition 2.2.7. *By a fuzzy number on $\mathbb{R}$, we mean a mapping $\mu : \mathbb{R} \to [0,1]$ with the following properties*:

(i) *μ is upper semi-continuous.*

(ii) *μ is fuzzy convex, that is, $\mu(\lambda x + (1-\lambda)y) \geq \min(\mu(x), \mu(y))$ for all $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$.*

(iii) *μ is normal, that is, there exists $x_0 \in \mathbb{R}$ such that $\mu(x_0) = 1$.*

(iv) *Closure of the support of μ is compact, that is, $cl\{x \in \mathbb{R} : \mu(x) > 0\}$ is compact in $\mathbb{R}$.*

Remark 2.2.8. *Let $\mathbb{E}$ (or $\mathbb{E}^1$) denote the set of all fuzzy numbers on $\mathbb{R}$. $\mathbb{R}$ is embedded $\mathbb{E}$. It follows easily from the fact that $\forall c \in \mathbb{R}$, the mapping $c \to \hat{c}$ from $\mathbb{R}$ into $\mathbb{E}$ is injective, where $\hat{c} \in \mathbb{E}$ is defined by*

$$
\hat{c}(x) = \begin{cases} 1 & \text{if } x = c \\ 0 & \text{if } x \neq c. \end{cases}
$$

It can be easily shown that for every $\mu \in \mathbb{E}$, for $\alpha \in [0,1]$, the α-level set of μ is closed and bounded interval which is denoted by $[\mu]_{\alpha} = [\underline{\mu^{\alpha}}, \overline{\mu^{\alpha}}]$, where $\underline{\mu^{\alpha}}$, $\overline{\mu^{\alpha}}$ are lower and upper $\alpha-$cuts of μ, respectively.

A fuzzy number μ in $\mathbb{E}$ in its parametric form is represented by $(\underline{\mu^{\alpha}}, \overline{\mu^{\alpha}}, \alpha)$. The following lemma due to Goetschel Jr and Voxman (1986) provides a characterization of fuzzy numbers on $\mathbb{R}$.

Lemma 2.2.9. *(Goetschel Jr and Voxman (1986)) Assume that $I = [0, 1]$, and $a : I \to \mathbb{R}$ and $b : I \to \mathbb{R}$ satisfy the conditions:*

(a) $a : I \to \mathbb{R}$ *is a bounded increasing function.*

(b) $b : I \to \mathbb{R}$ *is a bounded decreasing function.*

(c) $a(1) \leq b(1)$.

(d) For $0 < k \leq 1$, $\lim_{\alpha \to k^-} a(\alpha) = a(k)$ *and* $\lim_{\alpha \to k^-} b(\alpha) = b(k)$.

(e) $\lim_{\alpha \to 0^+} a(\alpha) = a(0)$ *and* $\lim_{\alpha \to 0^+} b(\alpha) = b(0)$.

Then $\mu : \mathbb{R} \to I$ defined by

$$\mu(x) = \sup\{\alpha | a(\alpha) \leq x \leq b(\alpha)\}$$

is a fuzzy number with parametrization given by $\{(a(\alpha), b(\alpha), \alpha) | 0 \leq \alpha \leq 1\}$. Moreover, if $\mu : \mathbb{R} \to I$ is a fuzzy number with parametrization given by $\{(a(\alpha), b(\alpha), \alpha) | 0 \leq \alpha \leq 1\}$, where $(a(\alpha))$, $b(\alpha)$ are the lower and upper α-cuts of μ, then functions $a(\alpha)$ and $b(\alpha)$ satisfy conditions $(a) - (e)$.

The following representation theorem characterizes the space $\mathbb{E}^n$, the space of fuzzy numbers defined on $\mathbb{R}^n$. Let $P_k(\mathbb{R}^n)$ denote the set of all nonempty compact and convex subsets of $\mathbb{R}^n$.

Theorem 2.2.10. *(Negoita and Ralescu (1975)) If $u \in \mathbb{E}^n$, then we have*

(i) $[u]_\alpha \in \mathbb{P}_k(\mathbb{R}^n)$, *for all* $0 \leq \alpha \leq 1$.

(ii) $[u]_{\alpha_2} \subset [u]_{\alpha_1}$, *for all* $0 \leq \alpha_1 \leq \alpha_2 \leq 1$.

(iii) If $\{\alpha_k\}$ is a non-decreasing sequence converging to $\alpha > 0$, then $[u]_\alpha = \cap_{k \geq 1} [u]_{\alpha_k}$.

Conversely, if $\{A_\alpha\}$ is a family of subsets of $\mathbb{R}^n$ satisfying $(i) - (iii)$, then there exists an $u \in \mathbb{E}^n$ such that $[u]_\alpha = A_\alpha$ for $0 < \alpha \leq 1$ and $[u]_0 = \overline{\cup_{0 < \alpha \leq 1} A_\alpha} \subset A_0$.

Definition 2.2.11. *Given two fuzzy numbers $X_0 = (X_{01}, \ldots, X_{0n})$, $X_1 = (X_{11}, \ldots, X_{1n})$ in $(\mathbb{E}^1)^n$, we say $X_0 \leq X_1$ if $\mu_{X_{0i}}(\cdot) \leq \mu_{X_{1i}}(\cdot)$, $1 \leq i \leq n$. If $\mu_{X_{0i}}(\cdot) = \mu_{X_{1i}}(\cdot)$, for $1 \leq i \leq n$ then we say $X_0 = X_1$ otherwise $X_0 \neq X_1$. $\mu_{X_{0i}}$ and $\mu_{X_{1i}}$ denotes the membership functions for X_{0i} and X_{1i}, respectively.*

We now introduce the well known fundamental arithmetic operations on fuzzy numbers on $\mathbb{R}$. Arithmetic operations on fuzzy numbers are defined through the arithmetic operations on their α-cuts. Since we know that α-cuts of fuzzy numbers on $\mathbb{R}$ are closed and bounded intervals, therefore we first need to define the arithmetic operations on closed and bounded intervals of $\mathbb{R}$. Let $*$ denote any of the arithmetic operations: addition $(+)$, subtraction $(-)$, multiplication $(\times)$, or division $(/)$. Let $[a, b]$ and $[c, d]$ be any closed and bounded intervals of $\mathbb{R}$. Then we have

$$[a, b] * [c, d] = \{x * y | a \leq x \leq b \text{ and } c \leq y \leq d\}.$$

This yields

(i) $[a, b] + [c, d] = [a + c, b + d]$.

(ii) $[a, b] - [c, d] = [a - d, b - c]$.

(iii) $[a, b] \times [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$.

(iv) $[a, b]/[c, d] = [a, b] \times [\frac{1}{d}, \frac{1}{c}]$ provided $0 \notin [c, d]$.

Let A and B be fuzzy numbers in $\mathbb{E}^1$, then the α-cuts of the fuzzy number $A * B$ in $\mathbb{E}^1$ is defined by

$$(A * B)^\alpha = A^\alpha * B^\alpha.$$

By using decomposition Theorem 2.2.6, the fuzzy number $(A * B)$ is defined as

$$A * B = \bigcup_{\alpha \in [0,1]} {}_\alpha(A * B).$$

Definition 2.2.12. *(Extension Principle) The extension principle for fuzzy sets is in essence a basic identity which allows the domain of the definition of a mapping or a* *relation to be extended from points in X to fuzzy subsets of X.(cf. Zadeh (1975)). More specifically, let $f : X \longrightarrow Y$ be a given function. Then the Zadeh's extension principle leads to the extension $f^* : \mathbb{F}(X) \longrightarrow \mathbb{F}(Y)$ of f defined as follows:*

$$f^*(A)(y) = \sup_{x|f(x)=y} A(x) \qquad \forall A \in \mathbb{F}(X).$$

The function f^* is called fuzzy extension of f. Further, suppose $f : X_1 \times X_2 \times \ldots \times X_n \longrightarrow Y$. Then the extension principle leads to the following definition of $f^* : \mathbb{F}(X_1) \times \mathbb{F}(X_2) \times \ldots \mathbb{F}(X_n) \longrightarrow \mathbb{F}(Y)$. For $A_i \in \mathbb{F}(X_i)$, $1 \leq i \leq n$, we have:

$$f^*(A_1, A_2, \ldots, A_n)(y) = \sup_{x=(x_1,x_2,\ldots,x_n)|f(x)=y} (A_1(x_1) \wedge A_2(x_2) \wedge \ldots \wedge A_n(x_n)), \quad (2.7)$$

where $\wedge$ denotes the standard fuzzy intersection operator.

By using above extension principle, the fuzzy arithmetic operations can be easily obtained. This can be seen as follows. Let $*$ denote any of the standard arithmetic operations $+, -, \times, or/$. Let A and B be any fuzzy sets on $\mathbb{R}$. Define a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $f(x, y) = x * y$ for $x, y \in \mathbb{R}$. Then the application of extension principle on f leads to the following definition of arithmetic operations of fuzzy sets on $\mathbb{R}$

$$(A * B)(z) = \sup_{(x,y)|x*y=z} (A(x) \wedge B(y).$$

For example, by replacing $*$ with $+$, we have the following definition of $A + B$.

$$(A + B)(x) = \sup_{x=y+z} \min(A(y), B(z)), x \in \mathbb{R}.$$

Given any $A \in \mathbb{F}(\mathbb{R})$, scalar multiplication on fuzzy sets of X is defined as follows (see Dubois and Prade (1982)):

$$(\beta A)(x) = \begin{cases} u(\frac{x}{\beta})\} & \text{if } \beta \in \mathbb{R} \setminus \{0\} \\ \tilde{0} & \text{if } \beta = 0, \end{cases}$$

where $\tilde{0} \in \mathbb{F}(\mathbb{R})$ is given by

$$\tilde{0}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$$

Thus, it is clear that there is a linear structure on the space $\mathbb{E}^1$, the space $\mathbb{E}^1$ is not a vector space. However, the fuzzy numbers $\mathbb{E}^1$ with addition and multiplication as defined above is a convex cone. The fuzzy numbers can be isometrically embedded into a suitable normed linear space (cf. Puri and Ralescu (1983)).

Definition 2.2.13. *The Hausdorff metric d_H on $\mathbb{P}_k(\mathbb{R}^n)$ is defined as*

$$d_H(A, B) = \inf\{\epsilon \,|\, A \subset N(B, \epsilon) \, and \, B \subset N(A, \epsilon)\}, \tag{2.8}$$

where A, $B \in \mathbb{P}_k(\mathbb{R}^n)$ and $N(A, \epsilon) = \{x \in \mathbb{R} \,|\, |x - y| < \epsilon \, for \, some \; y \in A\}$, $N(B, \epsilon)$ is similarly defined.

The Housdorff distance $d_H(A, B)$ between A and B in $\mathbb{P}_k(\mathbb{R}^n)$ is also defined by

$$d_H(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} |a - b|, \sup_{b \in B} \inf_{a \in A} |a - b|\}.$$

It is well known that $(\mathbb{P}_k(\mathbb{R}), d_H)$ is complete metric space (cf. Kaleva (1987), Puri and Ralescu (1983)). We can define a metric on E^n by using Hausdorff metric. Define $D : \mathbb{E}^n \times \mathbb{E}^n \to \mathbb{R}^+ \cup \{0\}$ by

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} d_H(u^\alpha, v^\alpha),$$

where d_H is the Hausdorff metric defined on $\mathbb{P}_k(\mathbb{R}^n)$.

Definition 2.2.14. *A mapping $F : I = [a, b] \to \mathbb{E}^n$ is continuous at $t_0 \in I$ if given any $\epsilon > 0$ there exists a $\delta > 0$, such that*

$$D(F(t), F(t_0)) < \epsilon \quad whenever \quad |t - t_0| < \delta.$$

Now we will introduce the notion of differentiability for fuzzy functions. There are various notions of differentiability for fuzzy set valued function. Few important among them are H-derivatives or Puri and Ralescu Derivative, Seikkala Derivative, and Kandel-Friedman-Ming derivatives (cf. Buckley and Feuring (2000)). We shall now provide the formal definition for Puri and Ralescu Derivative and Seikkala Derivative.

Definition 2.2.15. *(Puri and Ralescu Derivative) First, we define the Hukuhara difference between two fuzzy numbers A and B defined on $\mathbb{R}^n$. If there exists a fuzzy number C on $\mathbb{R}^n$ such that $C + A = B$ then C is called Hukuhara difference between B and A and we write this as*

$$C = B \overset{*}{-} A.$$

A mapping $F : I = [a, b] \to \mathbb{E}^n$ is differentiable at $t_0 \in I$ if there exists a fuzzy number $\dot{F}(t_0) \in \mathbb{E}^n$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t_0 + h) \overset{*}{-} F(t_0)}{h}, \lim_{h \to 0^+} \frac{F(t_0) \overset{*}{-} F(t_0 - h)}{h},$$

exist and equal to $\dot{F}(t_0)$. Here the limits are taken in the metric $(\mathbb{E}^n, D)$ (cf. Puri and Ralescu (1983)).

Definition 2.2.16. *(Seikkala Derivative) Let $F : I = [a, b] \to \mathbb{E}^1$ be a given function. Suppose the parametric form of $F(t)$ is represented by*

$$F(t) = \{(F_1(t, \alpha), F_2(t, \alpha), \alpha) : \alpha \in [0, 1], t \in I\}.$$

The Seikkala (1987) derivative $\dot{F}(t)$ of $F(t)$ is defined by

$$\dot{F}(t) = \{(\dot{F}_1(t, \alpha), \dot{F}_2(t, \alpha), \alpha) : \alpha \in [0, 1], t \in I\}, \quad (2.9)$$

provided that the above equation represents a fuzzy number in the parametric form (cf. Seikkala (1987)).

Definition 2.2.17. *A set valued function $F : [0, T] \to \mathbb{P}_k(\mathbb{R}^n)$ is said to be measurable if it satisfies any one of the following equivalent conditions:*

(i) For all $x \in \mathbb{R}^n$, the mapping $t \to d_{F(t)}(x) = \inf_{u \in F(t)} \| x - u \|$ is measurable.

(ii) $Gr(F) = \{(t,x) \in [0,T] \times \mathbb{R}^n : x \in F(t)\} \in \sigma \times \beta(\mathbb{R}^n)$, *where* σ *and* $\beta(\mathbb{R}^n)$ *are the Boral* σ*-field of* $[0,T]$ *and* $\mathbb{R}^n$*, respectively.*

(iii) There exists a sequence $\{f_n(\cdot)\}$ *of measurable functions such that* $F(t) = \overline{\cup_{n\geq 1} f_n(t)}$ *for all* $t \in [0,T]$ *(Casting representation).*

Let $F : [0,T] \to P_k(\mathbb{R}^n)$, we denote by S_F^1, the set of all measurable selections of $F(\cdot)$ that belong to Lebesgue-Bochner space $L^1_{\mathbb{R}^n}$, that is,

$$S_F^1 = \left\{ f(\cdot) \in L^1_{\mathbb{R}^n} : f(t) \in F(t), a.e. \right\},$$

and the Aumann's integral is presented as follows:

$$(A) \int_0^T F(t)dt = \left\{ \int_0^T f(t)dt, f(\cdot) \in S_F^1 \right\}.$$

We say that $F : [0,T] \to P_k(\mathbb{R}^n)$ is integrably bounded if it is measurable and there exists a function $h : [0,T] \to \mathbb{R}$, and $h \in L^1_{\mathbb{R}}$ such that $\| x \| \leq h(t)$, $x \in F(t)$. If F is integrably bounded then $\int_0^T F(t)dt$ is compact in $\mathbb{R}^n$.

We say a fuzzy set valued mapping $F : [0,T] \to \mathbb{E}^n$ is integrably bounded if $F_0(t) = [F(t)]_0$ is integrably bounded.

Definition 2.2.18. *Let* $F : [0,T] \to \mathbb{E}^n$ *be a fuzzy integrably bounded mapping. Let* $F_\alpha(t) = [F(t)]_\alpha$*, then the fuzzy integral of* F *over* $[0,T]$*, denoted by* $\int_0^T F(t)dt$*, is defined levelwise by*:

$$\left[\int_0^T F(t)dt \right]_\alpha = (A) \int_0^T F_\alpha(t)dt, 0 < \alpha \leq 1.$$

This integral is well defined (see (Puri and Ralescu, 1986, Theorem 3.1)). Furthermore, various properties of the fuzzy integral are discussed in Kaleva (1987).

Let $F : [0,T] \times \mathbb{E}^n \to \mathbb{E}^n$. Consider the following fuzzy initial value problem:

$$\dot{x} = F(t,x), x(0) = x_0. \tag{2.10}$$

Definition 2.2.19. *A mapping* $x : [0,T] \to \mathbb{E}^n$ *is a fuzzy weak solution to* (2.10) *if it is* *continuous and satisfies the following integral equation*:

$$x(t) = x_0 + \int_0^t F(s, x(s))ds, \forall t \in [0, T].$$

If F is continuous then the weak solution also satisfies (2.10) and we call it strong fuzzy solution to (2.10) as mentioned in (Kaleva, 1987, Lemma 6.1).

CHAPTER 3

Evolution of Solutions of Fuzzy Dynamical Systems

3.1 Introduction

The theory of fuzzy differential equations has been developed in late seventies and early eighties. The term "fuzzy differential equation" was first used by Kandel and Byatt in 1978 in a short note (cf. Kandel and Byatt (1978)). An extended version of this short note was published in 1980 by the same authors (cf. Kandel and Byatt (1980)). Since then so many authors have substantially contributed towards the theory of fuzzy differential equations. In the theory of fuzzy differential equations many approaches are suggested to define a solution of a fuzzy differential equation. Among them Hukuhara approach or H-differentiability approach (cf. Dubey and George (2012b), Kaleva (2006), Khastan et al. (2011), Nieto (1999), Puri and Ralescu (1983), Wu et al. (1996)), differential inclusion approach (cf. Diamond (1999), Hüllermeier (1997), Mizukoshi et al. (2007)), extension principle approach (cf. Buckley and Feuring (2000)) and the level-wise or α-cut approach due to Seikkala (1987) are the most referred in the literature. Although several authors have contributed in theoretical and applied fields for fuzzy differential equations considering the H-derivatives, still in some cases the approach suffers certain disadvantages since the diameter $diam(x(t))$ of the solution $x(t)$ of an FDE is unbounded as time t increases. These disadvantages have been overcome up to certain extent by introducing a more general definition of H-derivative namely generalized H-differentiability (cf. Bede and Stefanini (2013)), fuzzy lateral H-derivative (cf. Chalco-Cano and Román-Flores (2008)). Recently, Bede and Gal (2005) and Chalco-Cano and Román-Flores (2008) studied the fuzzy differential equations by considering fuzzy lateral H-derivative. Furthermore, Chalco-Cano and Román-Flores (2009), investigated the relationships with fuzzy differential inclusions and fuzzy differential equations considering generalized H-derivative.

In recent years, many authors studied the qualitative properties of solutions of FDEs by considering the levelwise approach. Unlike other approaches levelwise approach allows to translate a system of fuzzy differential equations in to a system of ordinary differential equations corresponding to the end points of α-cuts of the states. This translation of FDE in to a system of ODEs allows to borrow many of the standard concepts from the established theory of ordinary differential equations. Due to this reason levelwise approach has attracted the attention of many researchers (for example, Diamond and Kloeden (1994), Ghazanfari et al. (2012), Xu et al. (2010, 2007), Dubey and George (2013d)) to study and analyze the various aspects of solutions of fuzzy differential equations.

The early work in the direction of levelwise approach is initiated by Seikkala (1987). Seikkala (1987) has shown that by using the Zadeh's extension principle, the evolution of fuzzy initial value problem $\dot{x}(t) = f(t, x(t)), x(t_0) = x_0$, where $f : R^+ \times \mathbb{R} \to \mathbb{R}$ is measurable in t and continuous in x, and x_0 is a fuzzy number on $\mathbb{R}$, is expressed by a system of 2-ordinary differential equations corresponding to the end points of the α-cuts of x. However, the proof is missing. Following the idea of Seikkala (1987), Xu et al. (2007) has considered the linear fuzzy initial value problem of the type $\dot{x}(t) = Ax(t), x(t_0) = X_0$, where A is $n \times n$ real matrix and X_0 is an n-vector of fuzzy numbers on $\mathbb{R}$. In Xu et al. (2007), it is stated (see Lemma 3.2 of Xu et al. (2007)) that the evolution of aforesaid system can be described by a system of $2n$-ordinary differential equations corresponding to the end points of the α-cuts of x. Similar results are stated in Dubey and George (2012a) for the systems of the type $\dot{x}(t) = Ax(t) + Bu(t), x(t_0) = X_0$, where A and B are $n \times n$ and $n \times m$ real matrices, respectively and $X_0 \in (\mathbb{E}^1)^n$, $u(t) \in (\mathbb{E}^1)^m$.

These results motivate us to investigate the solutions of a general nonlinear system with fuzzy initial conditions and fuzzy inputs by using the levelwise approach. To the best of our knowledge, we feel that a detailed proof of the results describing the evolution of solutions of such systems in terms of α-cuts is missing in the literature. We have shown with a complete proof that the solutions of such systems are described by a system of ordinary differential equations with crisp initial conditions and crisp inputs corresponding to the end points of the α-cuts of the fuzzy states. The novelty of our results lies in their applicability to nonlinear systems with uncertain but fuzzily modelled initial conditions and control variables. Our results can also be regarded as the generalizations of some of the results in the literature (for example, Seikkala (1987), Xu et al. (2007)).

The organization of the chapter is as follows: In Section 3.2, we investigate the solutions of an n-dimensional nonlinear fuzzy initial value problem

$$\dot{x}(t) = f(t, x(t), u(t)), x(t_0) = x_0, \tag{3.1}$$

where $f : R^+ \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a nonlinear function which is measurable in t and is continuous in x and u. The fuzzy initial condition x_0 is an n-vector of fuzzy numbers on $\mathbb{R}$, that is, $x_0 \in (\mathbb{E}^1)^n$, and the fuzzy input $u(t)$ is an m-vector of fuzzy numbers on $\mathbb{R}$, that is, $u(t) \in (\mathbb{E}^1)^m$. By using the extension principle, we prove that the evolution of system (3.1) can be described by a system of $2n$-ordinary differential equations corresponding to the end points of the α-cuts of x. In Section 3.3, we discuss the existence and uniqueness of the solutions of time-varying linear systems with fuzzy initial conditions and fuzzy inputs; we also provide a closed form formula for the solutions of such systems by using the complex number representation of the α-cuts of the fuzzy states and transition matrix techniques. In Section 3.4, we discuss the existence and uniqueness of the solutions of nonlinear fuzzy initial value problem of the type

$$\dot{x}(t) = f(t, x(t)), x(t_0) = x_0 \in (\mathbb{E}^1)^n, t \geq t_0 \geq 0,$$

where $f : T \times (\mathbb{E}^1)^n \rightarrow (\mathbb{E}^1)^n$ is continuous. Finally, Section 3.5 concludes the chapter.

3.2 Solutions of Nonlinear Systems of ODEs with Fuzzy Initial Conditions and Fuzzy Inputs

In this section, we investigate the solutions of nonlinear systems of ODEs with fuzzy initial conditions and fuzzy inputs. Let $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a given function such that $f(t, x, u)$ is measurable with respect to t and continuous with respect to x and

u. That is, the system we consider is formulated as:

$$\dot{x}(t) = f(t, x(t), u(t)),\ x(t_0) = x_0 \in (\mathbb{E}^1)^n,\ u(t) \in (\mathbb{E}^1)^m,\ t \geq t_0 \geq 0. \tag{3.2}$$

In the following theorem, we will show that by using the extension principle the evolution of the solutions of system (3.2) is described by a $2n$-differential equations with crisp initial conditions and crisp inputs corresponding to the end points of the α-cuts of x.

Theorem 3.2.1. *Let $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ and $[x_k(t)]_\alpha = [\underline{x_k^\alpha}(t), \overline{x_k^\alpha}(t)]$ be the α-cut of $x_k(t)$ for $1 \leq k \leq n$. $u(t) = (u_1(t), u_2(t), \ldots, u_m(t))$ and $[u_k(t)]_\alpha = [\underline{u_k^\alpha}(t), \overline{u_k^\alpha}(t)]$ be the α-cut of $u_k(t)$ for $1 \leq k \leq m$. The evolution of system (3.2) is described by the following set of $2n$-levelwise differential equations corresponding to the end points of the α-cuts of $x(t)$. That is, for each $\alpha \in [0,1]$ and $1 \leq k \leq n$, we have:*

$$\begin{cases} \dot{\underline{x_k^\alpha}}(t) = \min(f_k(t,z,w) : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)], w_j \in [\underline{u_j^\alpha}(t), \overline{u_j^\alpha}(t)]) \\ \dot{\overline{x_k^\alpha}}(t) = \max(f_k(t,z,w) : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)], w_j \in [\underline{u_j^\alpha}(t), \overline{u_j^\alpha}(t)]) \\ \underline{x_k^\alpha}(t_0) = \underline{x_{0k}^\alpha} \\ \overline{x_k^\alpha}(t_0) = \overline{x_{0k}^\alpha}, \end{cases}$$

where $z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$, $w = (w_1, w_2, \ldots, w_m) \in \mathbb{R}^m$ and f_k is the k^{th} component of f.

Proof. Since $x(t) \in (\mathbb{E}^1)^n$, $u(t) \in (\mathbb{E}^1)^m$ for all $t \geq t_0$, therefore, $f_k(\cdot,\cdot,\cdot)$ is extended by using the extension principle. The fuzzy extension $f_k^*(\cdot,\cdot,\cdot) : \mathbb{R}^+ \times (\mathbb{E}^1)^n \times (\mathbb{E}^1)^m \longrightarrow \mathbb{E}^1$ of f_k is defined as follows:

$$f_k^*(t, x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t))(y) = \sup_{(\tau,\nu)=(\tau_1,\ldots,\tau_n,\nu_1,\ldots,\nu_m) | f_k(t,\tau,\nu)=y} \{\min(x_1(t)(\tau_1), \ldots, x_n(t)(\tau_n), u_1(t)(\nu_1), \ldots, u_m(t)(\nu_m))\}.$$

Fuzzification of system (3.2) yields for each $1 \leq k \leq n$

$$\dot{x_k}(t) = f_k^*(t, x(t), u(t)),\ x_k(t_0) = x_{0k}, \tag{3.3}$$

in which f_k^* is the fuzzy extension of f_k.

By taking the α-cuts from both sides of (3.3), we have:

$$[\dot{x_k}(t)]_\alpha = [f_k^*(t, x(t), u(t))]_\alpha, [x_k(t_0)]_\alpha = [x_{0k}]_\alpha. \tag{3.4}$$

From Theorem 5.2 of Kaleva (1987) we have that:

$$[\dot{x_k}(t)]_\alpha = [\underline{\dot{x_k^\alpha}}(t), \overline{\dot{x_k^\alpha}}(t)]. \tag{3.5}$$

We shall now show the following:

$$\begin{aligned}[f_k^*(t, x(t), u(t))]_\alpha = [\,&\min(f_k(t, z, w) : z_i \in [x_i(t)]_\alpha, w_j \in [u_j(t)]_\alpha), \\ &\max(f_k(t, z, w) : z_i \in [x_i(t)]_\alpha, w_j \in [u_j(t)]_\alpha)].\end{aligned} \tag{3.6}$$

Let $s \in [\min(f_k(t, z, w) : z_i \in [x_i(t)]_\alpha, w_j \in [u_j(t)]_\alpha), \max(f_k(t, z, w) : z_i \in [x_i(t)]_\alpha, w_j \in [u_j(t)]_\alpha)]$. Since f_k is continuous with respect to x and u, therefore there exist $\tau_i \in [x_i(t)]_\alpha$ for $1 \leq i \leq n$ and $\nu_j \in [u_j(t)]_\alpha$ for $1 \leq j \leq m$, such that $f_k(t, \tau_1, \ldots, \tau_n, \nu_1, \ldots, \nu_m) = s$. From which it follows that:

$$\begin{aligned}f_k^*(t, x(t), u(t))(s) = \sup_{(\tau,\nu)=(\tau_1,\ldots,\tau_n,\nu_1,\ldots,\nu_m) \mid f_k(t,\tau,\nu)=s} &\{\min(x_1(t)(\tau_1), \ldots, \\ &x_n(t)(\tau_n), u_1(t)(\nu_1), \ldots, u_m(t)(\nu_m))\} \geq \alpha.\end{aligned}$$

Hence $s \in [f_k^*(t, x(t), u(t))]_\alpha$.

Conversely, assume that $s \in [f_k^*(t, x(t), u(t))]_\alpha$, which implies that

$$f_k^*(t, x(t), u(t))(s) \geq \alpha.$$

Define

$$\begin{aligned}\rho := \sup_{(\tau,\nu)=(\tau_1,\ldots,\tau_n,\nu_1,\ldots,\nu_m) \mid f_k(t,\tau,\nu)=s} &\{\min(x_1(t)(\tau_1), \ldots, x_n(t)(\tau_n), u_1(t)(\nu_1), \\ &\ldots, u_m(t)(\nu_m))\}.\end{aligned} \tag{3.7}$$

Clearly $\rho \geq \alpha$. From Eq. (3.7) it follows that, there exists a sequence $\{\varsigma^p\}$, where

$$\varsigma^p := (\tau_1^p, \ldots, \tau_n^p, \nu_1^p, \ldots, \nu_m^p)$$

with $f_k(t, \tau_1^p, \ldots, \tau_n^p, \nu_1^p, \ldots, \nu_m^p) = s$, such that

$$\rho = \sup_{\varsigma=(\tau_1,\ldots,\tau_n,\nu_1,\ldots,\nu_m)\in\{\varsigma^p\}} \{\min(x_1(t)(\tau_1), \ldots, x_n(t)(\tau_n), u_1(t)(\nu_1), \ldots, u_m(t)(\nu_m))\}. \tag{3.8}$$

From Eq. (3.8) it follows that:

$$\limsup_{p\to\infty} x_1(t)(\tau_1^p) \geq \limsup_{p\to\infty} (\min(x_1(t)(\tau_1^p), \ldots, x_n(t)(\tau_n^p), u_1(t)(\nu_1^p), \ldots, u_m(t)(\nu_m^p))) \geq \alpha.$$

Without loss of generality it can be assumed that $\{\tau_1^p\} \in [x_1(t)]_0$ for otherwise the entries in the sequence $\{\tau_1^p\}$ which do not belong to $[x_1(t)]_0$ can be simply ignored. Since $[x_1(t)]_0$ is compact, therefore there exists a subsequence $\{\tau_1^{p_r^{(1)}}\}$ of $\{\tau_1^p\}$(indeed a subsequence of $\{\tau_1^p\}$) such that $\tau_1^{p_r^{(1)}} \longrightarrow \tau_1^*$ as $r \to \infty$ for some $\tau_1^* \in \mathbb{R}$ and

$$\limsup_{r\to\infty} x_1(t)(\tau_1^{p_r^{(1)}}) = \limsup_{p\to\infty} x_1(t)(\tau_1^p) \geq \alpha.$$

Now by using upper semi-continuity of $x_1(t)$ we have:

$$x_1(t)(\tau_1^*) \geq \limsup_{r\to\infty} x_1(t)(\tau_1^{p_r^{(1)}}) \geq \alpha.$$

Therefore, we have:

$$\tau_1^* \in [x_1(t)]_\alpha.$$

Since $\{\varsigma^{p_r^{(1)}}\}$ is a subsequence of $\{\varsigma^p\}$, therefore we have:

$$\rho = \sup_{\varsigma=(\tau_1,\ldots,\tau_n,\nu_1,\ldots,\nu_m)\in\{\varsigma^{p_r^{(1)}}\}} \{\min(x_1(t)(\tau_1), \ldots, x_n(t)(\tau_n), u_1(t)(\nu_1), \ldots, u_m(t)(\nu_m))\}.$$

Using the same arguments as for $x_1(t)$, we will have a subsequence $\{\varsigma^{p_r^{(2)}}\}$ of $\{\varsigma^{p_r^{(1)}}\}$ such that $\tau_2^{p_r^{(2)}} \longrightarrow \tau_2^*$ as $r \longrightarrow \infty$ and $\tau_2^* \in [x_2(t)]_\alpha$.

Continuing in the same fashion, we will have nested subsequences of $\{\varsigma^{p_r^{(1)}}\}$, and points $\tau_i^* \in \mathbb{R}$ for $1 \leq i \leq n$, and $\nu_j^* \in \mathbb{R}$ for $1 \leq j \leq m$. That is,

$$\{\varsigma^{p_r^{(n+m)}}\} \subset \ldots \{\varsigma^{p_r^{(n+1)}}\} \subset \{\varsigma^{p_r^{(n)}}\} \subset \{\varsigma^{p_r^{(n-1)}}\} \ldots \subset \{\varsigma^{p_r^{(2)}}\} \subset \{\varsigma^{p_r^{(1)}}\},$$

with the property that for $1 \leq i \leq n$,

$$\tau_i^{p_r^{(i)}} \longrightarrow \tau_i^* \text{ as } r \longrightarrow \infty, \text{ and } \tau_i^* \in [x_i(t)]_\alpha,$$

and for $1 \leq j \leq m$,

$$\nu_j^{p_r^{(n+j)}} \longrightarrow \nu_j^* \text{ as } r \longrightarrow \infty, \text{ and } \nu_j^* \in [u_j(t)]_\alpha.$$

Clearly $f_k(t, \tau_1^{p_r^{(n+m)}}, \ldots, \tau_n^{p_r^{(n+m)}}, \nu_1^{p_r^{(n+m)}}, \ldots, \nu_m^{p_r^{(n+m)}}) = s$ for all $r \in \mathbb{N}$. Also, for $1 \leq i \leq n$, $\tau_i^{p_r^{(n+m)}} \longrightarrow \tau_i^*$ as $r \longrightarrow \infty$ and for $1 \leq j \leq m$, $\nu_j^{p_r^{(n+m)}} \longrightarrow \nu_j^*$ as $r \longrightarrow \infty$. Now by continuity of f_k we have:

$$f_k(t, \tau_1^{p_r^{(m+n)}}, \ldots, \tau_n^{p_r^{(m+n)}}, \nu_1^{p_r^{(m+n)}}, \ldots, \nu_m^{p_r^{(m+n)}}) \longrightarrow f_k(t, \tau_1^*, \ldots, \tau_n^*, \nu_1^*, \ldots, \nu_m^*).$$

Hence it follows that $f_k(t, \tau_1^*, \ldots, \tau_n^*, \nu_1^*, \ldots, \nu_m^*) = s$ with $\tau_i^* \in [x_i(t)]_\alpha$ for $1 \leq i \leq n$, and $\nu_j^* \in [u_j(t)]_\alpha$ for $1 \leq j \leq m$. Thus, we must have:

$$\begin{aligned} s \in [&\min(f_k(t, z, w) : z_i \in [x_i(t)]_\alpha, w_j \in [u_j(t)]_\alpha), \\ &\qquad \max(f_k(t, z, w) : z_i \in [x_i(t)]_\alpha, w_j \in [u_j(t)]_\alpha)]. \end{aligned}$$

Hence we have established (3.6). Now the proof of the theorem follows from equations (3.4), (3.5) and (3.6). □

In Lemma 1 of Dubey and George (2012a), a similar result is employed in order to compute the controllable initial fuzzy states for the linear systems of the form $\dot{x}(t) = Ax(t) + Bu(t), x(t_0) = x_0 \in (\mathbb{E}^1)^n, u(t) \in (\mathbb{E}^1)^m$, $t > t_0 \in \mathbb{R}^+$, where A and B are real matrices of size $n \times n$, $n \times m$, respectively.

Corollary 3.2.2 (Dubey and George (2012a)). *Let $f(t, x(t), u(t)) = Ax(t) + Bu(t)$ in* (3.2)*, where A, B are real matrices of size $n \times n$ and $n \times m$, respectively. Then, the evolution of system* (3.2) *is described by a $2n$-differential equations corresponding to the end points of the α-cuts of $x(t)$. That is, for $1 \leq k \leq n$,*

$$\begin{cases} \dot{\underline{x_k^\alpha}}(t) = \min((Az + Bw)_k : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)], w_j \in [\underline{u_j^\alpha}(t), \overline{u_j^\alpha}(t)]) \\ \dot{\overline{x_k^\alpha}}(t) = \max((Az + Bw)_k : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)], w_j \in [\underline{u_j^\alpha}(t), \overline{u_j^\alpha}(t)]) \\ \underline{x_k^\alpha}(t_0) = \underline{x_{0k}^\alpha} \\ \overline{x_k^\alpha}(t_0) = \overline{x_{0k}^\alpha}, \end{cases}$$

in which $(Az + Bw)_k$ denotes the k^{th} row of $(Az + Bw)$.

If $f : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be such that $f(t, x)$ is measurable with respect to t, and continuous with respect to x. Then the evolution of fuzzy initial value problem

$$\dot{x}(t) = f(t, x(t)), x(t_0) = x_0 \in (\mathbb{E}^1)^n, t \geq t_0 \geq 0 \tag{3.9}$$

will be given by a $2n$-differential equations corresponding to the end points of the α-cuts of x. Thus, we have the following theorem.

Theorem 3.2.3. *Let $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$, and $[x_k(t)]_\alpha = [\underline{x_k^\alpha}(t), \overline{x_k^\alpha}(t)]$ be the α-cut of $x_k(t)$ for $1 \leq k \leq n$. The evolution of system* (3.9) *is described by the following set of $2n$-levelwise equations corresponding to the end points of the α-cuts of $x(t)$. That is, for each $\alpha \in [0, 1]$ and $1 \leq k \leq n$, we have:*

$$\begin{cases} \dot{\underline{x_k^\alpha}}(t) = \min(f_k(t, z) : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)]) \\ \dot{\overline{x_k^\alpha}}(t) = \max(f_k(t, z) : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)]) \\ \underline{x_k^\alpha}(t_0) = \underline{x_{0k}^\alpha} \\ \overline{x_k^\alpha}(t_0) = \overline{x_{0k}^\alpha}, \end{cases}$$

in which $z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$ and f_k is the k^{th} component of f.

Proof. The proof follows along the similar lines of the proof of Theorem 3.2.1. □

As a special case, a similar result for the linear time-invariant systems $\dot{x}(t) = Ax(t), x(t_0) = x_0 \in (\mathbb{E}^1)^n$, where A is $n \times n$ real matrix, $t \geq t_0 \in \mathbb{R}^+$, is used by Xu et al. (2007).

Corollary 3.2.4 (Xu et al. (2007))**.** *Let $f(t, x(t)) = Ax(t)$, where A is a real matrix of size $n \times n$. Then, the evolution of the system* (3.9) *is described by a $2n$-differential equations corresponding to the end points of α-cuts of $x(t)$. That is, for $1 \leq k \leq n$,*

$$\begin{cases} \dot{\underline{x_k^\alpha}}(t) = \min((Az)_k : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)]) \\ \dot{\overline{x_k^\alpha}}(t) = \max((Az)_k : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)]) \\ \underline{x_k^\alpha}(t_0) = \underline{x_{0k}^\alpha} \\ \overline{x_k^\alpha}(t_0) = \overline{x_{0k}^\alpha}, \end{cases}$$

where $(Az)_k$ denotes the k^{th} row of (Az).

The following result due to Seikkala (1987) follows immediately from Theorem 3.2.3. Seikkala (1987) considered the fuzzy initial value problem of the type

$$\dot{x}(t) = f(t, x(t)), x(t_0) = x_0 \in \mathbb{E}, t \geq t_0 \geq 0, \tag{3.10}$$

where $f : R^+ \times \mathbb{R} \to \mathbb{R}$ is measurable in t and continuous in x.

Corollary 3.2.5 (Seikkala (1987))**.** *The evolution of the system* (3.10) *is described by a 2-differential equations corresponding to the end points of α-cuts of $x(t)$. That is,*

$$\begin{cases} \dot{\underline{x^\alpha}}(t) = \min(f(t, z) : z \in [\underline{x^\alpha}(t), \overline{x^\alpha}(t)]) \\ \dot{\overline{x^\alpha}}(t) = \max(f(t, z) : z \in [\underline{x^\alpha}(t), \overline{x^\alpha}(t)]) \\ \underline{x^\alpha}(t_0) = \underline{x_0^\alpha} \\ \overline{x^\alpha}(t_0) = \overline{x_0^\alpha}. \end{cases}$$

3.2.1 Numerical Example

In this section, we provide an example to explain the fuzzification of nonlinear system of ODEs and the evolution of solutions as prescribed by Theorem 3.2.1.

Example 3.2.6. *Consider the following system of nonlinear ODEs with fuzzy initial conditions and fuzzy inputs*:

$$\begin{pmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{pmatrix} = \begin{pmatrix} -x_2^2(t) + cos(t) + u_1(t) \\ -x_1^2(t) + sin(t) + u_2(t) \end{pmatrix}. \tag{3.11}$$

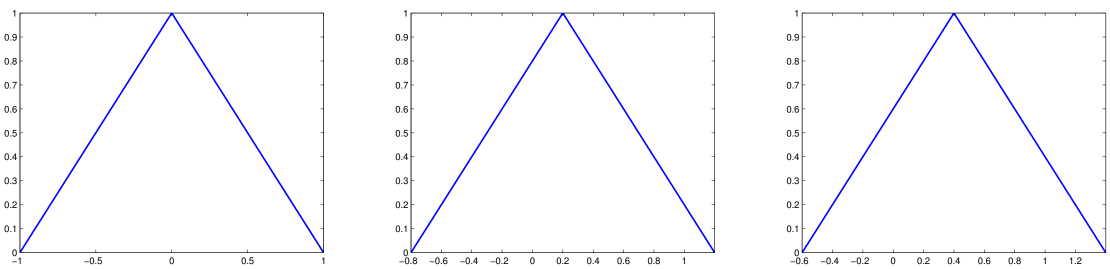
Let the initial membership functions $x_1(0) = \mu_1(s)$ *and* $x_2(0) = \mu_2(s)$ *be defined as below*:

$$\mu_1(s) = \begin{cases} s & 0 \leq s \leq 1 \\ 2 - s & 1 \leq s \leq 2 \end{cases}, \qquad \mu_2(s) = \begin{cases} 2s & 0 \leq s \leq 1/2 \\ 2 - 2s & 1/2 \leq s \leq 1. \end{cases}$$

Taking the fuzzy inputs $u_1(t), u_2(t) \in \mathbb{E}^1$ as defined by the following fuzzy numbers:

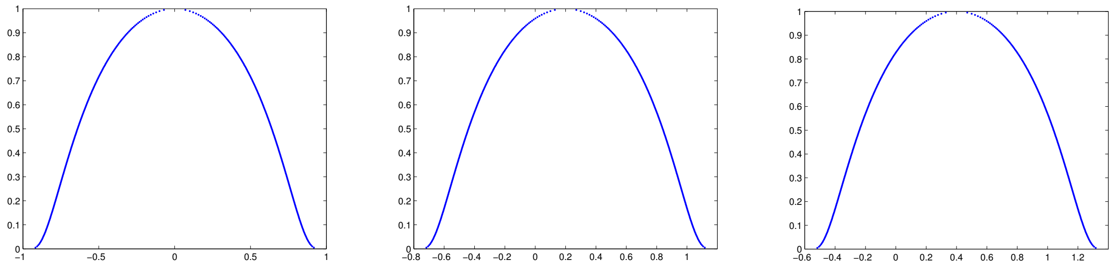
$$u_1(t)(s) = \begin{cases} s - t + 1, & t - 1 \leq s \leq t \\ -s + t + 1, & t \leq s \leq t + 1 \end{cases}, u_2(t)(s) = \begin{cases} e^{1 - \frac{1}{1 - |s-t|^2}}, & \mid s - t \mid \leq 1 \\ 0, & \mid s - t \mid \geq 1. \end{cases}$$

The input functions $u_1(t)$ and $u_2(t)$ at various time instants are shown in Figure 3.1 and Figure 3.2, respectively. The evolution of the solutions of system (3.11) is given by the



(a) u_1 at time $t = 0$ (b) u_1 at time $t = \cdot 2$ (c) u_1 at time $t = \cdot 4$

Figure 3.1: Input function u_1 at various time instants



(a) u_2 at time $t = 0$ (b) u_2 at time $t = \cdot 2$ (c) u_2 at time $t = \cdot 4$

Figure 3.2: Input function u_2 at various time instants

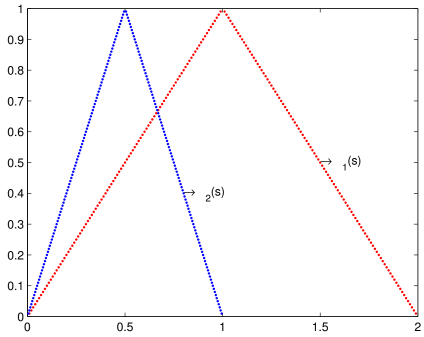
following levelwise decomposed differential equations:

$$\begin{cases} \dot{\underline{x_1^\alpha}}(t) = -\overline{x_2^\alpha}^2(t) + cos(t) + (t + \alpha - 1) \\ \dot{\underline{x_2^\alpha}}(t) = -\overline{x_1^\alpha}^2(t) + sin(t) + (t - \mid (1 - \dfrac{1}{log\left(\frac{e}{\alpha}\right)})^{\frac{1}{2}} \mid) \\ \dot{\overline{x_1^\alpha}}(t) = -\underline{x_2^\alpha}^2(t) + cos(t) + (t + 1 - \alpha) \\ \dot{\overline{x_2^\alpha}}(t) = -\underline{x_1^\alpha}^2(t) + sin(t) + (t + \mid (1 - \dfrac{1}{log\left(\frac{e}{\alpha}\right)})^{\frac{1}{2}} \mid) \end{cases},$$

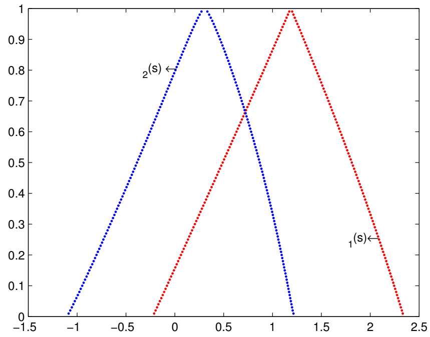
with the initial condition

$$[\underline{x_1^\alpha}(0), \underline{x_2^\alpha}(0), \overline{x_1^\alpha}(0), \overline{x_2^\alpha}(0)]^T = [\alpha, \alpha/2, 2 - \alpha, 1 - (\alpha/2)]^T$$

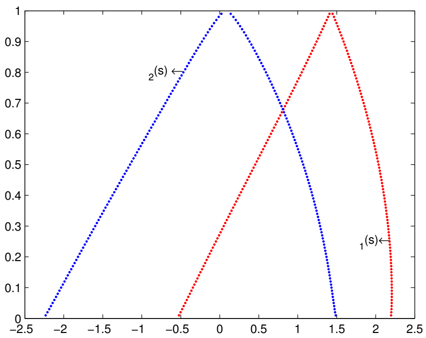
and α varies in $[0, 1]$. The propagated fuzzy states at time $t = \cdot 2$ and $t = \cdot 4$ starting from the initial fuzzy state at time $t = 0$, are shown in Figure 3.3. It must be noted that the existence of the solutions of Eq.(3.11) is guaranteed only in the interval $[0, T]$ for some $T > 0$ (see Kaleva (1987), Song and Wu (2000)). In this example, the solutions of system (3.11) exist in the interval [0,T], where $\cdot 4 < T < \cdot 6$. For $t > T$ the solutions cease to exist as shown in Figure 3.4. It is clear from the Figure 3.4a that $x_1(\cdot 6) \notin \mathbb{E}^1$. Similarly at $t = 1$, $x_1(1) \notin \mathbb{E}^1$ and $x_2(1) \notin \mathbb{E}^1$ as indicated in Figure 3.4b. The cross sections of level sets at various levels (α-values) for x_1 and x_2 during time interval [0,1] are shown in Figure 3.5 and Figure 3.6, respectively.



(a) at time $t = 0$



(b) at time $t = \cdot 2$



(c) at time $t = \cdot 4$

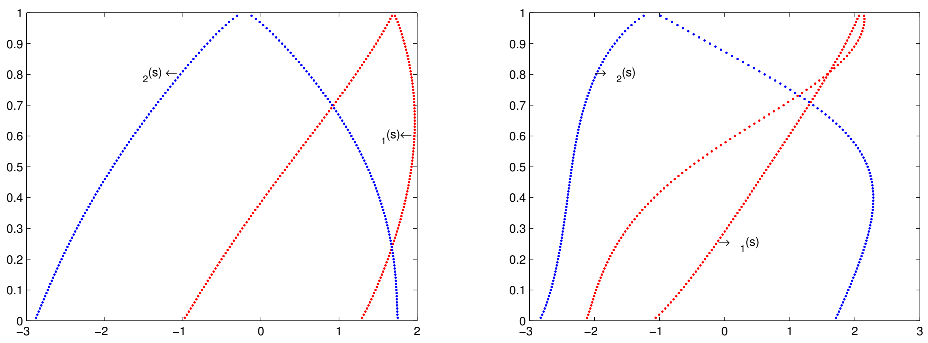
Figure 3.3: Propagation of membership functions for x_1 and x_2

3.3 Solutions of Linear Systems with Fuzzy Initial Conditions and Fuzzy Inputs

In this section, we investigate the solutions of the fuzzy differential dynamical systems of the type:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ x(t_0) = X_0, t_1 \geq t \geq t_0, \end{cases} \tag{3.12}$$

where $A(\cdot) \in C([t_0, t_1]; \mathbb{R}^{n\times n}), B(\cdot) \in C([t_0, t_1]; \mathbb{R}^{n\times m}), X_0 \in (\mathbb{E}^1)^n = \underbrace{\mathbb{E}^1 \times \ldots \times \mathbb{E}^1}_{n-times}$, $t_0 \in \mathbb{R}^+$, the input $u(t) \in (\mathbb{E}^1)^m$ for each $t \in [t_0, t_1]$, moreover $u(\cdot)$ is assumed to be integrable in $[t_0, t_1]$. We discuss the existence and uniqueness of the solutions of the system (3.12) and obtain a closed form formula for the solutions of the system (3.12) by using the levelwise approach and the transition matrix technique.



(a) at time $t = \cdot 6$ (b) at time $t = 1$

Figure 3.4: x_1 and x_2 at $t = \cdot 6$ and $t = 1$

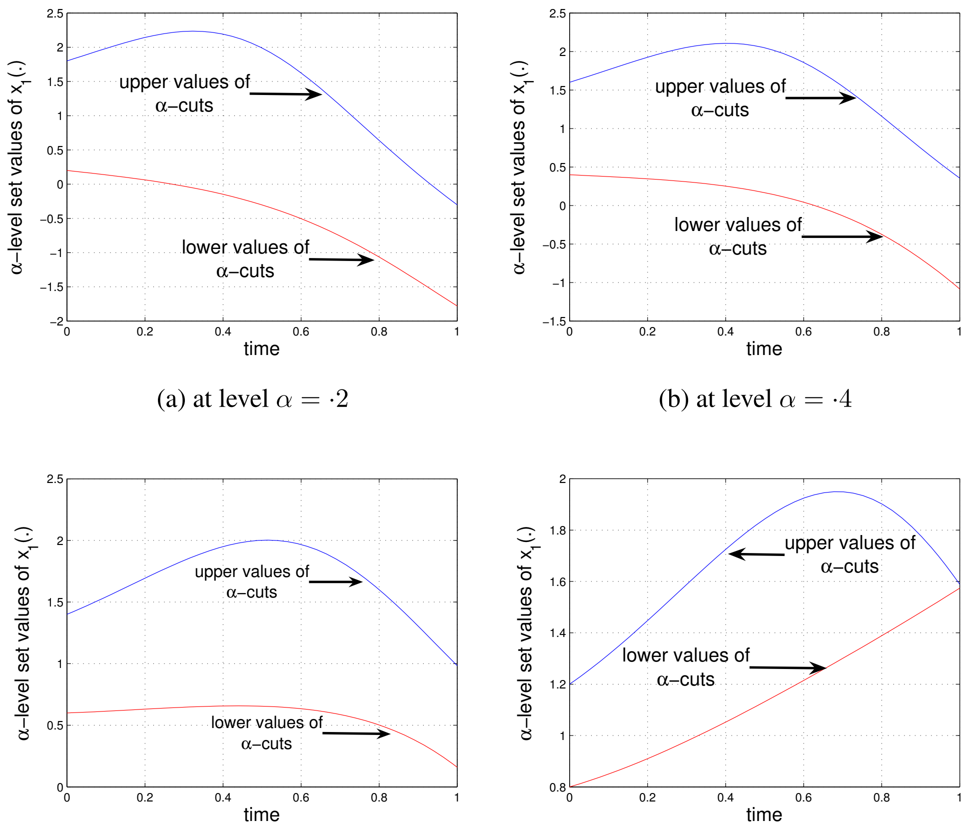
Many authors have investigated the solutions of n-dimensional fuzzy initial value problem by using the levelwise approach and provided techniques for the computation of the solutions. For our purpose, we refer to the work of Buckley and Feuring (2001), Fard and Ghal-Eh (2011), Xu et al. (2010) and Xu et al. (2007).In particular, Xu et al. (2007) have described the solutions of the systems of type $\dot{x}(t) = Ax(t), x(0) = X_0 \in (\mathbb{E}^1)^n$(time-invariant system) by using complex number representation of the α-level sets of fuzzy system following the idea given in Pearson (1997). In our work, we extend some of the results of Xu et al. (2007) to the time-varying systems of type (3.12). The following lemma describes the evolution of solutions of the system (3.12).

Lemma 3.3.1. *Let $x_k^\alpha(t) = [\underline{x_k^\alpha}(t), \overline{x_k^\alpha}(t)]$ be the α-cut of $x_k(t)$ for $1 \leq k \leq n$. The evolution of the system* (3.12) *is described by the following 2n-differential equations, corresponding to the end points of $\alpha-$cuts: For $1 \leq k \leq n$,*

$$\begin{cases} \dot{\underline{x_k^\alpha}}(t) = \min((A(t)z + B(t)w)_k : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)], w_j \in [\underline{u_j^\alpha}(t), \overline{u_j^\alpha}(t)]) \\ \dot{\overline{x_k^\alpha}}(t) = \max((A(t)z + B(t)w)_k : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)], w_j \in [\underline{u_j^\alpha}(t), \overline{u_j^\alpha}(t)]) \\ \underline{x_k^\alpha}(t_0) = \underline{X_{0k}^\alpha} \\ \overline{x_k^\alpha}(t_0) = \overline{X_{0k}^\alpha}, \end{cases} \tag{3.13}$$

where $(A(t)z+B(t)w)_k = \Sigma_{j=1}^n a_{kj}(t)z_j + \Sigma_{j=1}^m b_{kj}(t)w_j$ is the k^{th} row of $A(t)z+B(t)w$.

Proof. The proof follows immediately from the Theorem 3.2.1. □



(a) at level $\alpha = \cdot 2$ (b) at level $\alpha = \cdot 4$

(c) at level $\alpha = \cdot 6$ (d) at level $\alpha = \cdot 8$

Figure 3.5: Cross sections of the level sets for $x_1(\cdot)$ during time interval [0,1]

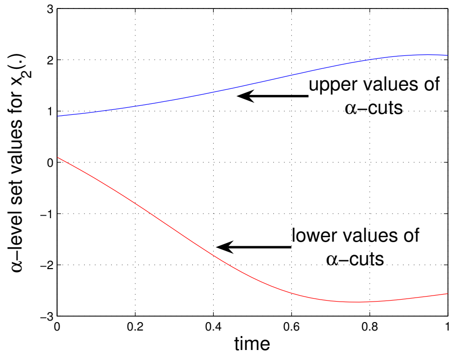
By using the above lemma, we have the following theorem that provides a compact formula for the solutions of system (3.12).

Theorem 3.3.2. *The unique solution of system* (3.12) *is characterized by the unique solution of the following system:*

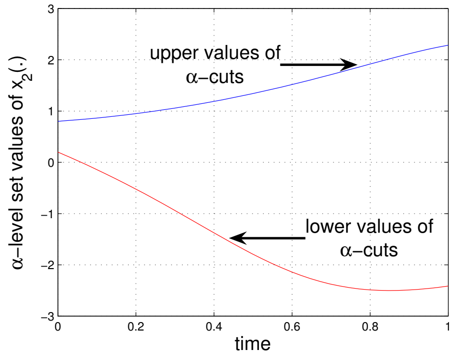
$$\begin{cases} \dot{\underline{x}}^{\alpha}(t) + i\dot{\overline{x}}^{\alpha}(t) = M(t)(\underline{x}^{\alpha}(t) + i\overline{x}^{\alpha}(t)) + N(t)(\underline{u}^{\alpha}(t) + i\overline{u}^{\alpha}(t)) \\ \underline{x}^{\alpha}(t_0) + i\overline{x}^{\alpha}(t_0) = \underline{X}_0^{\alpha} + i\overline{X}_0^{\alpha}, \end{cases} \tag{3.14}$$

where $i^2 = -1, \alpha \in [0,1]$ *and entries of* $M(t)$ *and* $N(t)$ *are obtained from that of* $A(t)$ *and* $B(t)$ *as follows:* $m_{jk}(t) = ha_{jk}(t), 1 \leq j,k \leq n$, *and* $n_{jk}(t) = hb_{jk}(t), 1 \leq j \leq n, 1 \leq k \leq m$, *i.e.,* $M(t) = hA(t)$ *and* $N(t) = hB(t)$, *where for each* $s \in \mathbb{R}$

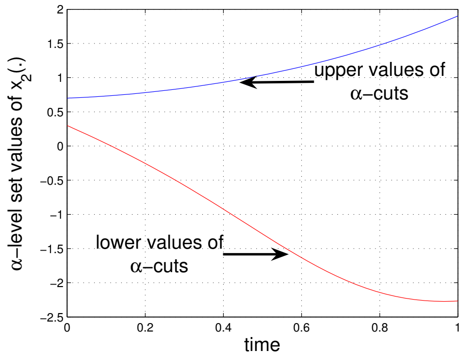
$$hs = \begin{cases} h_1 s & s \geq 0 \\ h_2 s & s < 0 \end{cases}$$



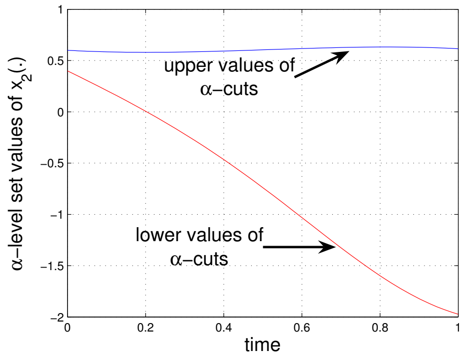
(a) at level $\alpha = \cdot 2$



(b) at level $\alpha = \cdot 4$



(c) at level $\alpha = \cdot 6$



(d) at level $\alpha = \cdot 8$

Figure 3.6: Cross sections of the level sets for $x_2(\cdot)$ during time interval [0,1]

in which $h_1 s : \mathbb{C} \to \mathbb{C}$ *is the identity operator and is defined as*

$$h_1 s : a + ib \to s(a + ib)$$

and $h_2 s : \mathbb{C} \to \mathbb{C}$ *corresponds to a flip about the diagonal in the complex plane and is defined as*

$$h_2 s : a + ib \to s(b + ia)$$

for every $a + ib \in \mathbb{C}$.

Proof. It is clear that $h_2^2 = h_1$ and $h_2 h_1 = h_1 h_2$. It can be easily seen that $\{sh_j = h_j s : j = 1, 2\}$. Given any real matrix $P = [p_{ij}]$ and $k \geq 1$ the operator matrix $h^k P$ is obtained by applying the operator h^k to the entries on P, that is, $h^k P = [h^k p_{ij}]$. It can be easily shown that system (3.13) and system (3.14) are equivalent, by comparing the real and imaginary parts from both sides of (3.14). So the evolution of system (3.12) will be given by system (3.14). Now we propose a solution to system (3.14) as follows:

$$\underline{x^{\alpha}}(t)+i\overline{x^{\alpha}}(t) = \Phi^{M}(t,t_0)(\underline{X_0^{\alpha}}+i\overline{X_0^{\alpha}})+\int_{t_0}^{t}\Phi^{M}(t,\tau)N(\tau)(\underline{u^{\alpha}}(\tau)+i\overline{u^{\alpha}}(\tau))\mathrm{d}\tau, \quad (3.15)$$

where $\Phi^{M}(t,t_0)$ is expressed by a slightly modified Peano-Baker type series:

$$\Phi^{M}(t,t_0) = I^{*} + h\int_{t_0}^{t}A(\tau)\mathrm{d}\tau + h^2\int_{t_0}^{t}A(\tau)\int_{t_0}^{\tau}A(s)\mathrm{d}s\mathrm{d}\tau + \quad (3.16)$$

$$h^3\int_{t_0}^{t}A(\tau)\int_{t_0}^{\tau}A(s)\int_{t_0}^{s}A(w)\mathrm{d}w\mathrm{d}s\mathrm{d}\tau\ldots,$$

where $I^{*} = hI$ and I is $n\times n$ identity matrix. It can be shown that the above series is uniformly bounded in $[t_0,t_1]$ by using the fact that h_1, h_2 are bounded linear operators and $A\in C([t_0,t_1];\mathbb{R}^{n\times n})$. Thus, $\Phi^{M}(t,t_0)$ is well defined. Now from (3.16) it follows that

$$\frac{d}{dt}\Phi^{M}(t,t_0) = M(t)\Phi^{M}(t,t_0). \quad (3.17)$$

Further, it can be shown that $\Phi^{M}(\cdot,\cdot)$ satisfies the semigroup property, i.e.,

$$\Phi^{M}(t,t_0)\Phi^{M}(t_0,t_1) = \Phi^{M}(t,t_1). \quad (3.18)$$

In equation (3.18) by taking $t_1 = t$, we have:

$$\Phi^{M}(t,t_0)\Phi^{M}(t_0,t) = I^{*}. \quad (3.19)$$

By differentiating Eq. (3.15) and using Eq. (3.17), we have

$$\begin{aligned}\frac{d}{dt}(\underline{x^{\alpha}}(t)+i\overline{x^{\alpha}}(t)) =& M(t)\Phi^{M}(t,t_0)(\underline{X_0^{\alpha}}+i\overline{X_0^{\alpha}}) + N(t)(\underline{u^{\alpha}}(t)+i\overline{u^{\alpha}}(t))\\ &+ M(t)\int_{t_0}^{t}\Phi^{M}(t,\tau)N(\tau)(\underline{u^{\alpha}}(\tau)+i\overline{u^{\alpha}}(\tau))\mathrm{d}\tau \quad (3.20)\\ \frac{d}{dt}(\underline{x^{\alpha}}(t)+i\overline{x^{\alpha}}(t)) =& M(t)(\underline{x^{\alpha}}(t)+i\overline{x^{\alpha}}(t)) + N(t)(\underline{u^{\alpha}}(t)+i\overline{u^{\alpha}}(t)). \quad (3.21)\end{aligned}$$

From (3.15) and the definition of $\Phi^{M}(t,t_0)$, it is obvious that $\underline{x^{\alpha}}(t_0)+i\overline{x^{\alpha}}(t_0) = \underline{X_0^{\alpha}} + i\overline{X_0^{\alpha}}$. Hence (3.15) defines a solution of system (3.14). We claim that solution given by (3.15) is unique. Let $\underline{y}^{\alpha}(t) + i\overline{y^{\alpha}}(t)$ be another solution of (3.14). Define a function by:

$$\underline{z}^{\alpha}(t)+i\overline{z^{\alpha}}(t) = \Phi^{M}(t_0,t)\left(\underline{y}^{\alpha}(t) + i\overline{y^{\alpha}}(t) - \int_{t_0}^{t} \Phi^{M}(t,\tau)N(\tau)(\underline{u}^{\alpha}(\tau) + i\overline{u^{\alpha}}(\tau))\mathrm{d}\tau\right). \tag{3.22}$$

Differentiating Eq. (3.19), we have $\frac{d}{dt}\Phi^{M}(t_0,t) = -\Phi^{M}(t_0,t)M(t)$. Now differentiating (3.22), it follows that

$$\frac{d}{dt}(\underline{z}^{\alpha}(t) + i\overline{z^{\alpha}}(t)) = 0.$$

Thus, $\underline{z}^{\alpha}(t) + i\overline{z^{\alpha}}(t) = constant$. Setting $t = t_0$ in (3.22) yields $\underline{z}^{\alpha}(t_0) + i\overline{z^{\alpha}}(t_0) = \underline{X_0^{\alpha}} + i\overline{X_0^{\alpha}}$. Thus,

$$\underline{z}^{\alpha}(t) + i\overline{z^{\alpha}}(t) = \underline{X_0^{\alpha}} + i\overline{X_0^{\alpha}} \tag{3.23}$$

for all $t \in [t_0, t_1]$. Substituting (3.23) in (3.22) we have

$$\begin{aligned}\underline{y}^{\alpha}(t) + i\overline{y^{\alpha}}(t) &= \Phi^{M}(t,t_0)(\underline{X_0^{\alpha}} + i\overline{X_0^{\alpha}}) + \int_{t_0}^{t} \Phi^{M}(t,\tau)N(\tau)(\underline{u}^{\alpha}(\tau) + i\overline{u^{\alpha}}(\tau))\mathrm{d}\tau \\ &= \underline{x}^{\alpha}(t) + i\overline{x^{\alpha}}(t).\end{aligned}$$

Thus, the solution given by (3.15) is unique. □

Remark 3.3.3. *If the matrix $A(t)$, $\int_{t_0}^{t} A(\tau)d(\tau)$ commutes then $\Phi^{M}(t,t_0)$ is given by the following expression:*

$$\Phi^{M}(t,t_0) = \exp\left(h\int_{t_0}^{t} A(\tau)d\tau\right).$$

Remark 3.3.4. *If the matrix $A(\cdot)$ is constant matrix then $\Phi^{M}(t,t_0)$ can be given by the following expression:*

$$\Phi^{M}(t,t_0) = \exp\left(hA(t-t_0)\right).$$

A similar result for a special case of the system (3.12), that is when $A(t) = A \in \mathbb{R}^{n\times n}$ and $B(t) = 0$, is investigated in Xu et al. (2007). The following result of Xu et al. (2007) follows immediately from the Theorem 3.3.2.

Corollary 3.3.5. *(Xu et al. (2007)) Let A be $n \times n$ real matrix. Then for a given*

$x_0 \in (\mathbb{E}^1)^n$, the initial value problem

$$\begin{cases} \dot{x}(t) = Ax(t) \\ x(0) = x_0, t \geq 0 \end{cases}$$

has a unique solution given by

$$\begin{cases} \dot{\underline{x}^{\alpha}}(t) + i\dot{\overline{x^{\alpha}}}(t) = B(\underline{x}^{\alpha}(t) + i\overline{x^{\alpha}}(t)) \\ \underline{x}^{\alpha}(0) + i\overline{x^{\alpha}}(0) = \underline{x_0}^{\alpha} + i\overline{x_0}^{\alpha}, \end{cases}$$

where the elements of the matrix B are determined from that of A as follows:

$$b_{ij} = \begin{cases} h_1 a_{ij} & a_{ij} \geq 0 \\ h_2 a_{ij} & a_{ij} < 0, \end{cases}$$

in which h_1 is just the identity operation and h_2 corresponds to a flip about the diagonal in the complex plane, i.e., $\forall a + ib \in \mathbb{C}$,

$$\begin{aligned} h_1 : a + ib &\rightarrow a + ib, \\ h_2 : a + ib &\rightarrow b + ia. \end{aligned}$$

3.3.1 Numerical Examples

We will end this section by giving examples to demonstrate our results.

Example 3.3.6. *Consider the following time-varying fuzzy differential equation*:

$$\begin{pmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{pmatrix} = \begin{pmatrix} 2t & 1 \\ 0 & 2t \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} t.$$

The fuzzy initial condition $x(0) = (x_1(0), x_2(0))$, in which $x_1(0)$, $x_2(0)$ are given by following membership functions:

$$x_1(0)(s) = e^{1-\frac{1}{1-s^2}}, \qquad x_2(0)(s) = e^{1-\frac{1}{1-4s^2}}.$$

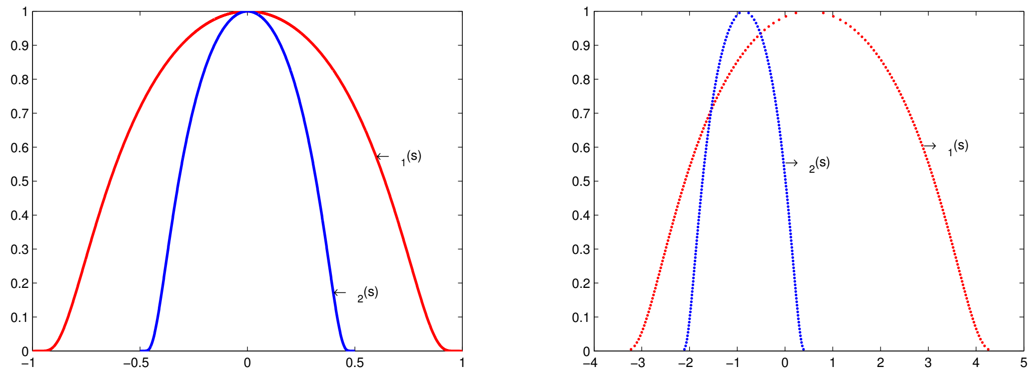
In this example $\Phi^M(t,0)$ *is given by the following expression*:

$$\Phi^M(t,0) = h\left(\exp\begin{pmatrix} t^2 & t \\ 0 & t^2 \end{pmatrix}\right).$$

The evolution of the system (3.25) *is described by the following levelwise decomposed system*

$$\begin{pmatrix} \dot{\underline{x}}_1^\alpha(t) \\ \dot{\underline{x}}_2^\alpha(t) \\ \dot{\overline{x}}_1^\alpha(t) \\ \dot{\overline{x}}_2^\alpha(t) \end{pmatrix} = \begin{pmatrix} 2t & 1 & 0 & 0 \\ 0 & 2t & 0 & 0 \\ 0 & 0 & 2t & 1 \\ 0 & 0 & 0 & 2t \end{pmatrix} \begin{pmatrix} \underline{x}_1^\alpha(t) \\ \underline{x}_2^\alpha(t) \\ \overline{x}_1^\alpha(t) \\ \overline{x}_2^\alpha(t) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} t \\ t \end{pmatrix}. \tag{3.24}$$

Using Theorem 3.3.2, the propagated membership functions at time $t = 1$ *starting from the initial membership functions at time* $t = 0$ *is shown in Figure 3.7.*



(a) at time $t = 0$

(b) at time $t = 1$

Figure 3.7: Fuzzy states for $x_1(\cdot)$ and $x_2(\cdot)$ at $t = 0$ and $t = 1$

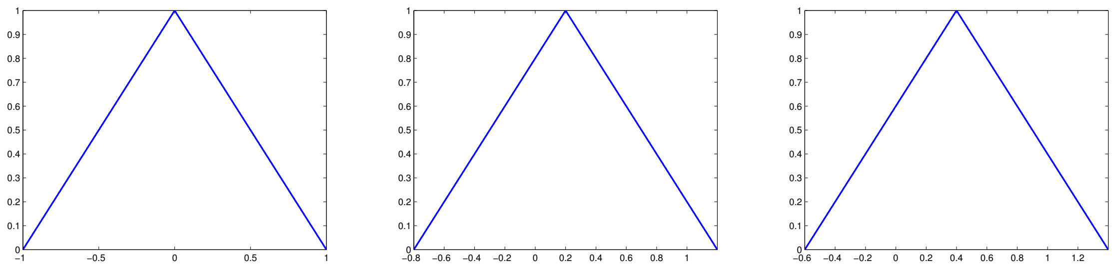
Example 3.3.7. *Consider the following time-varying fuzzy differential equation with fuzzy inputs and fuzzy initial condition*:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 2t & 1 \\ 0 & 2t \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} t \\ -t \end{pmatrix} u(t). \tag{3.25}$$

In this example the fuzzy initial condition is same as in the Example 3.3.6. The fuzzy *input* $u(t) \in \mathbb{E}^1$ *is defined by the following triangular fuzzy numbers*:

$$u(t)(s) = \begin{cases} s - t + 1, & t - 1 \leq s \leq t \\ -s + t + 1, & t \leq s \leq t + 1. \end{cases}$$

The input functions $u(t)$ *at various time instants are given in Figure 3.8. The evo-*



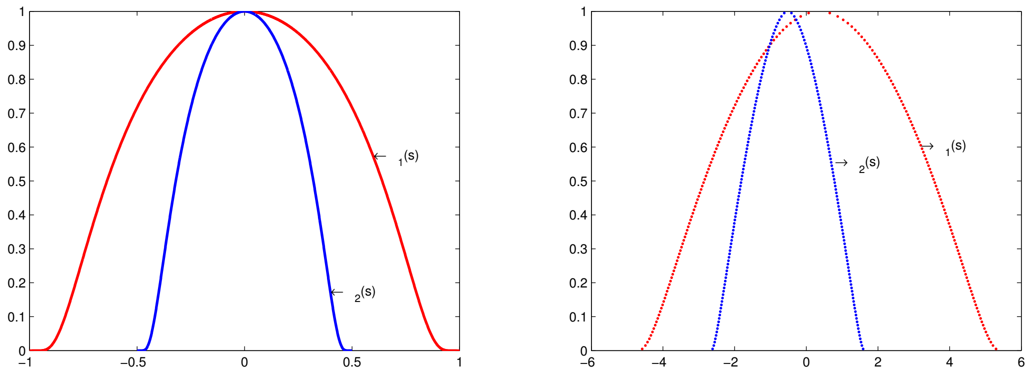
(a) u_1 at time $t = 0$ (b) u_1 at time $t = \cdot 2$ (c) u_1 at time $t = \cdot 4$

Figure 3.8: Input function u_1 at various time instants

lution of the system (3.25) *is described by the following levelwise decomposed system*

$$\begin{pmatrix} \dot{\underline{x}}_1^\alpha(t) \\ \dot{\underline{x}}_2^\alpha(t) \\ \dot{\overline{x}}_1^\alpha(t) \\ \dot{\overline{x}}_2^\alpha(t) \end{pmatrix} = \begin{pmatrix} 2t & 1 & 0 & 0 \\ 0 & 2t & 0 & 0 \\ 0 & 0 & 2t & 1 \\ 0 & 0 & 0 & 2t \end{pmatrix} \begin{pmatrix} \underline{x}_1^\alpha(t) \\ \underline{x}_2^\alpha(t) \\ \overline{x}_1^\alpha(t) \\ \overline{x}_2^\alpha(t) \end{pmatrix} + \begin{pmatrix} t & 0 \\ 0 & -t \\ 0 & t \\ -t & 0 \end{pmatrix} \begin{pmatrix} \underline{u}^\alpha(t) \\ \overline{u}^\alpha(t) \end{pmatrix}. \quad (3.26)$$

Fuzzy state of the solution at $t = 1$ *starting from the initial fuzzy state at time* $t = 0$ *is shown in Figure 3.9.*



(a) at time $t = 0$

(b) at time $t = 1$

Figure 3.9: Fuzzy states for $x_1(\cdot)$ and $x_2(\cdot)$ at $t = 0$ and $t = 1$

3.4 Existence and Uniqueness of a System of Fuzzy Differential Equations

In this section, we consider a general nonlinear fuzzy initial value problem of the type

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(t_0) = x_0 \in (\mathbb{E}^1)^n, t \in T, \end{cases} \tag{3.27}$$

where $f : T \times (\mathbb{E}^1)^n \longrightarrow (\mathbb{E}^1)^n$ is continuous and $T = [t_0, t_1](t_1 > t_0 \geq 0)$ be a compact interval of $\mathbb{R}$. We extend some of the results due to Kaleva (1987), which involves integrability and differentiability properties of the fuzzy set valued mappings, in a more general setting. We study the fuzzy differential equations in which the derivative of a fuzzy process is given by fuzzy n-vector valued mappings whose values are normal, convex, upper semi-continuous and compactly supported fuzzy sets in $\mathbb{R}$ and establish existence and uniqueness of a solution to such fuzzy differential equations. First, we discuss the measurability and integrability properties of the fuzzy-vector valued mappings. Subsequently, we establish existence and uniqueness of a solution to systems of type (3.27) by using Banach's fixed point theorem.

Let $D : \mathbb{E}^1 \times \mathbb{E}^1 \to R^+ \cup 0$, defined by the equation

$$D(u,v) = \sup_{0\leq\alpha\leq 1} d_H([u]_\alpha, [v]_\alpha), \tag{3.28}$$

where d_H is the Hausdorff metric defined on $\mathbb{P}_k(\mathbb{R})$ (refer Definition 2.2.13). It is easy to show that D is a metric space, and indeed (E^1, D) is complete metric space (cf. Puri and Ralescu (1983)). It is well known that

$$D(u+w, v+w) = D(u,v), \tag{3.29}$$

for all $u, v, w \in \mathbb{E}^1$. Define $D^* : (\mathbb{E}^1)^n \times (\mathbb{E}^1)^n \to \mathbb{R}^+ \cup 0$ by the equation

$$D^*(u,v) = \sup_{1\leq i\leq n} D(u_i, v_i), \tag{3.30}$$

where $u = (u_1, u_2, \ldots, u_n) \in (\mathbb{E}^1)^n$ and $v = [v_1, v_2, \ldots, v_n] \in (\mathbb{E}^1)^n$. By using equations (3.29) and (3.30), it is easy to show that

$$D^*(u+w, v+w) = D^*(u,v). \tag{3.31}$$

Further, it can be easily shown that

$$D^*(\lambda u, \lambda v) = |\lambda| D^*(u,v), \lambda \in \mathbb{R}. \tag{3.32}$$

3.4.1 Measurability and Integrability

Theorem 3.4.1. $((\mathbb{E}^1)^n, D^*)$ *is a complete metric space.*

Proof. Clearly $((\mathbb{E}^1)^n, D^*)$ is a metric space. Let $\{x^l\}$ be a cauchy sequence in $((\mathbb{E}^1)^n, D^*)$. Given $\epsilon > 0$, there exists $N(\epsilon) \in \mathbb{N}$ such that

$$D^*(x^l, x^m) \leq \epsilon \text{ for } l, m \geq N(\epsilon)\ . \tag{3.33}$$

From equation 3.33 we have,

$$\sup_{1\leq i\leq n} D(x_i^l, x_i^m) \leq \epsilon \text{ for } l, m \geq N(\epsilon)\ . \tag{3.34}$$

Therefore, for each $i \in \{1, 2, \ldots, n\}$, $\{x_i^l\}$ is a cauchy sequence in $(\mathbb{E}^1, D)$. Since $(\mathbb{E}^1, D)$ is complete, therefore there exists $x_i^* \in \mathbb{E}^1$ such that $x_i^l \longrightarrow x_i^*$ in $(\mathbb{E}^1, D)$ for each $1 \leq i \leq n$.

Define $x^* = (x_1^*, x_2^*, x_3^*, \ldots, x_n^*) \in \mathbb{E}^n$.

Claim : $x^l \longrightarrow x^*$ in $((\mathbb{E}^1)^n, D^*)$.

Since for each $i \in \{1, 2, \ldots, n\}$, $x_i^l \longrightarrow x_i^*$ therefore there exists $N_i(\epsilon) \in \mathbb{N}$ such that $D(x_i^l, x_i^*) < \epsilon$ for every $l \geq N_i(\epsilon)$. Define $N_* \in \mathbb{N}$ such that

$$N_* = \max(N_1(\epsilon), N_2(\epsilon), \ldots, N_n(\epsilon)).$$

Then it follows that for all $1 \leq i \leq n$

$$D(x_i^l, x_i^*) \leq \epsilon \text{ for } l \geq N_*. \tag{3.35}$$

From (3.35) it follows that

$$\sup_{1\leq i\leq n} D(x_i^l, x_i^*) \leq \epsilon \text{ for } l \geq N_*. \tag{3.36}$$

From the above equation it is immediate that $D^*(x^l, x^*) \leq \epsilon$ for $l \geq N_*$. Hence $((\mathbb{E}^1)^n, D^*)$ is complete. □

Let the mapping $F : T \to (\mathbb{E}^1)^n$ defined by

$$F(t) = (F_1(t), F_2(t), \ldots, F_n(t)).$$

Definition 3.4.2. *A mapping $F : T \to (\mathbb{E}^1)^n$ is strongly measurable if the set valued mappings $F_i^\alpha : T \to \mathbb{P}_k(\mathbb{R})$, where $\alpha \in [0, 1]$ and $1 \leq i \leq n$, defined by*

$$F_i^\alpha(t) = [F_i(t)]_\alpha$$

are Lebesgue measurable, when $\mathbb{P}_k(\mathbb{R})$ is endowed with the topology generated by the Hausdorff metric d_H.

Lemma 3.4.3. *If F is strongly measurable then it is measurable with respect to the topology generated by D^*.*

Proof. Let $\epsilon \geq 0$ and $u \in (\mathbb{E}^1)^n$. Define the set,

$$T_1 = \{t | D^*(F(t), u) \leq \epsilon\} \tag{3.37}$$

$$= \{t | \sup_{1 \leq i \leq n} D(F_i(t), u_i) \leq \epsilon.\} \tag{3.38}$$

$$= \cap_{i=1}^n \{t | \ D(F_i(t), u_i) \leq \epsilon\} \tag{3.39}$$

By Lemma 3.1 of Kaleva (1987), the set $\{t| \ D(F_i(t), u_i) \leq \epsilon\}$ is Lebesgue-measurable for each $1 \leq i \leq n$. Hence the set T_1, being the finite intersection of Lebesgue-measurable sets, is Lebesgue-measurable. Hence the proof of lemma. □

Lemma 3.4.4. *If $F : T \to (\mathbb{E}^1)^n$ is continuous with respect to the metric D^* then it is strongly measurable.*

Proof. Let $\epsilon \geq 0$ be arbitrary and $t_z \in T$ be any point. By the continuity of F, it follows that there exists a $\delta \geq 0$ such that

$$D^*(F(t), F(t_z)) < \epsilon \text{ whenever } |t - t_z| < \delta.$$

By definition of D^*, we have $d_H(F_i^\alpha(t), F_i^\alpha(t_0)) < \epsilon$ whenever $|t - t_0 <\leq \delta$ for all $1 \leq i \leq n$. That is, each $F_i^\alpha(\cdot)$ is continuous with respect to Housdorff metric. Therefore, given any open set $U \in \mathbb{P}_k(\mathbb{R})$, $F_i^{\alpha-1}(U)$ is open in $\mathbb{R}$ and hence measurable. □

Definition 3.4.5. *A mapping $F : T \to (\mathbb{E}^1)^n$ is called integrably bounded if there exists an integrable function $h(\cdot)$ such that $|x| \leq h(t)$ for all $x \in \cup_{i=1}^n F_i^0(t)$.*

Definition 3.4.6. *Let $F : T \to (\mathbb{E}^1)^n$. The integral of F over T, denoted by $\int_T F(t)dt$ and defined as follows:*

$$\int_T F(t)dt := \left(\int_T F_1(t)dt, \int_T F_2(t)dt, \ldots, \int_T F_n(t)dt \right),$$

in which for each $1 \leq i \leq n$, $\int_T F_i(t)dt$ is given by the following $\alpha-$cut representation.

$$\left[\int_T F_i(t)dt\right]_\alpha = \int_T F_i^\alpha(t)dt := \left[\int_T \underline{F_i^\alpha}(t), \int_T \overline{F_i^\alpha}(t)\right],$$

where $\underline{F_i^\alpha}(t)$, $\overline{F_i^\alpha}(t)$ denote the lower α-cut and upper α-cut of $F_i(t)$, respectively and $\alpha \in (0, 1]$.

A strongly measurable and integrably bounded mapping $F : T \to (\mathbb{E}^1)^n$ is said to be integrable over T if $\int_T F(t)dt \in (\mathbb{E}^1)^n$.

Theorem 3.4.7. *If $F : T \to (\mathbb{E}^1)^n$ is strongly measurable and integrably bounded then F is integrable.*

Proof. Since F is integrably bounded therefore each F_i is integrably bounded. Similarly each F_i is strongly measurable. Now the result is immediate from Theorem 4.1 of Kaleva (1987). □

Corollary 3.4.8. *If $F : T \to (\mathbb{E}^1)^n$ is continuous then it is integrable.*

Proof. By Lemma 3.4.4, F is strongly measurable. Let $\epsilon > 0$, and $t_0 \in T$. By continuity, there exist a $\delta > 0$ such that whenever $|t - t_0| \leq \delta$ we have $D^*(F(t), F(t_0)) \leq \epsilon$. Therefore for $1 \leq i \leq n$, F_i is continuous with respect to D metric, which means that $F_i^0(\cdot)$ is continuous with respect to Hausdorff metric. Since T is compact hence $\cup_{t \in T} F_i^0(t)$ is compact. Thus, each F_i is integrably bounded from which it follows that F is integrably bounded. Now the corollary follows from Theorem 3.4.7. □

The following theorem states the basic properties of fuzzy-integral.

Theorem 3.4.9. *Let $F, G : T \to (\mathbb{E}^1)^n$ be integrable and let $\lambda \in \mathbb{R}$. Then,*

(i) $\int_T F + G = \int_T F + \int_T G$

(ii) $\int_T \lambda F = \lambda \int_T F$

(iii) $D^*(F, G)$ *is integrable.*

(iv) $D^*(\int_T F, \int_T G) \leq \int_T D^*(F, G)$.

Proof. Since F is integrable which implies that for each $1 \leq i \leq n$, F_i is integrable. Now properties (i) and (ii) are obvious from Theorem 4.3 (i-ii) of Kaleva (1987). Now from Theorem 4.3(iii) of Kaleva (1987), we have that $D(F_i, G_i)$ is integrable. That is, there exist integrable functions $h_i(t)$ such that $D(F_i(t), G_i(t)) \leq h_i(t)$ for $1 \leq i \leq n$. Hence $D^*(F(t), G(t)) \leq \sum_{i=1}^{n} h_i(t)$, which means that $D^*(F, G)$ is integrable. This proves (iii). $D^*(\int_T F, \int_T G) = \sup_{1 \leq i \leq n} D(\int_T F_i, \int_T G_i) = D(\int_T F_k, \int_T G_k)$ for some $k \in \{1, 2, \ldots, n\}$. Now using Theorem 4.3 (iv) of Kaleva (1987), we have,

$$D^*(\int_T F, \int_T G) \leq \int_T D(F_k, G_k) \leq \int_T D^*(F, G).$$

Hence (iv) is proved. □

Corollary 3.4.10. *Let $F : T \to (\mathbb{E}^1)^n$ be integrable and $c \in T$. Then*

$$\int_{t_0}^{t_1} F = \int_{t_0}^{c} F + \int_{c}^{t_1} F$$

Proof. Define $F_1(t) = F(t)\chi_{[t_0,c]}$ and $F_2(t) = F(t)\chi_{[c,t_1]}$, where $\chi_{[t_0,c]}$, $\chi_{[c,t_1]}$ denote the characteristic functions on $[t_0, c]$ and $[c, t_1]$, respectively. By Theorem 3.4.9(i), it follows that

$$\int_T F = \int_T F_1 + F_2 = \int_T F_1 + \int_T F_2 = \int_{t_0}^{c} F + \int_{c}^{t_1} F.$$

Hence the corollary. □

Corollary 3.4.11. *If $F : T \to (\mathbb{E}^1)^n$ is continuous with respect to metric D^* then $G(t) = \int_{t_0}^{t} F$ is Lipschitz continuous on T.*

Proof. Let $s, t \in T$ be arbitrary, and $s > t$. Then by Corollary 3.4.10 and Eq. (3.31) we have

$$D^*\left(\int_{t_0}^{s} F, \int_{t_0}^{t} F\right) = D^*\left(\int_{t}^{s} F, \hat{\mathbf{0}}\right), \tag{3.40}$$

where $\hat{\mathbf{0}} = [\hat{0}, \hat{0}, \ldots, \hat{0}]^T \in (\mathbb{E}^1)^n$ in which $\hat{0}$ is defined as

$$\hat{0}(t) = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{if } t \neq 0. \end{cases}$$

Using property (iv) of Theorem 3.4.9 in Eq. (3.40),

$$D^*\left(\int_{t_0}^{s} F, \int_{t_0}^{t} F\right) \leq \int_{t}^{s} D^*\left(F, \hat{\mathbf{0}}\right), \tag{3.41}$$

Using continuity of F with respect to metric D^* we can show that the set $K := \cup_{1\leq i\leq n} \cup_{t\in T} F_i^0(t)$ is compact in $\mathbb{R}$. That is, there exists a constant $M > 0$ such that $|x| \leq M$ for all $x \in K$. Therefore, $D^*\left(F(t), \hat{\mathbf{0}}(t)\right) \leq M$ for all $t \in T$. Hence from Eq.(3.41), we have,

$$D^*(G(s), G(t)) \leq M(s-t).$$

Hence G is Lipschitz continuous. □

3.4.2 Existence and Uniqueness of a Fuzzy Initial Value Problem

Let $x, y \in (\mathbb{E}^1)^n$. If there exists a $z \in (\mathbb{E}^1)^n$ such that $x = y + z$ then we call z the $H-$difference of x and y, denoted by $x - y$.

Definition 3.4.12. *A mapping $F : T \to (\mathbb{E}^1)^n$ is differentiable at $t_0 \in T$ if there exists a $\dot{F}(t_0) := (\dot{F}_1(t_0), \dot{F}_2(t_0), \ldots, \dot{F}_n(t_0)) \in (\mathbb{E}^1)^n$ such that for each $1 \leq i \leq n$ the limits*

$$\lim_{h\to 0^+} \frac{F_i(t_0+h) - F_i(t_0)}{h}, \lim_{h\to 0^-} \frac{F_i(t_0+h) - F_i(t_0)}{h},$$

exist and equal to $\dot{F}_i(t_0)$. Here the limits are taken in metric $(\mathbb{E}^1, D)$.

Theorem 3.4.13. *If $F : T \to (\mathbb{E}^1)^n$ is differentiable then it is continuous.*

Proof. Let $t \in T$ and $h \geq 0$, then by triangle inequality for the metric D^* we have,

$$\begin{aligned} D^*(F(t+h), F(t)) &= D^*(F(t+h) - F(t), \hat{\mathbf{0}}) \\ &\leq hD^*((F(t+h) - F(t))/h, \dot{F}(t)) + hD^*(\dot{F}(t), \hat{\mathbf{0}}), \end{aligned}$$

where h is small enough so that the $H-$difference $F(t+h) - F(t)$ exists. Now by the differentiability of F the term on R.H.S goes to zero as $h \to 0^+$, that is $F(\cdot)$ is right continuous. Similarly left continuity of $F(\cdot)$ can be proved. □

The following two theorems follow immediately from Theorem 5.6 and Theorem 5.7 of Kaleva (1987), respectively.

Theorem 3.4.14. *Let $F : T \to (\mathbb{E}^1)^n$ be continuous. Then for all $t \in T$ the integral $G(t) = \int_{t_0}^{t} F$ is differentiable and $\dot{G}(t) = F(t)$.*

Theorem 3.4.15. *Let $F : T \to (\mathbb{E}^1)^n$ be differentiable and assume that the derivative $\dot{F}$ is integrable over T. Then for each $t \in T$ we have*

$$F(t) = F(a) + \int_{t_0}^{t} \dot{F}.$$

By using Theorems 3.4.13-3.4.15, the following lemma is immediate.

Lemma 3.4.16. *A mapping $x : T \to (\mathbb{E}^1)^n$ is a solution to the problem* (3.27) *if and only if it is continuous and satisfies the integral equation*

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds \tag{3.42}$$

for all $t \in T$.

If f is Lipschitz continuous then the problem (3.27) has a unique solution on T. The following theorem depicts this fact.

Theorem 3.4.17. *Let $f : T \times (\mathbb{E}^1)^n \to (\mathbb{E}^1)^n$ be continuous and assume that there exists a $k > 0$ such that $D^*(f(t, x), f(t, y)) \leq kD^*(x, y)$ for all $t \in T$ and $x, y \in E$. Then the problem* (3.27) *has a unique solution on T.*

Proof. Let $C(J, (\mathbb{E}^1)^n)$ denotes the set of all continuous mappings from J to $(\mathbb{E}^1)^n$, where J is an interval in R. We metricize $C(J, (\mathbb{E}^1)^n)$ by defining a new metric by

$$H(\phi, \psi) = \sup_{t \in J} D^*(\phi(t), \psi(t)). \tag{3.43}$$

Since $((\mathbb{E}^1)^n, D^*)$ is a complete metric space, therefore $C(J, (\mathbb{E}^1)^n)$ is also complete.

Now let $(t_2, y) \in T \times (\mathbb{E}^1)^n$ be arbitrary and $\eta > 0$ be such that $\eta k < 1$. We shall show that the initial value problem

$$\dot{x}(t) = f(t, x(t)), x(t_2) = y, \tag{3.44}$$

has a unique solution on the interval $I = [t_2, t_2 + \eta]$.

For $\xi \in C(I, (\mathbb{E}^1)^n)$ define $G(\xi)$ on I by the equation

$$G(\xi)(t) = y + \int_{t_2}^{t} f(s, \xi(s)ds).$$

Then by Corollary 3.4.11 $G(\xi) \in C(I, (\mathbb{E}^1)^n)$. Furthermore, by Theorem 3.4.9 and the Lipschitz continuity of f, we have

$$\begin{aligned} H(G(\phi), G(\psi)) &= \sup_{t\in I} D^*(\int_{t_2}^{t} f(s, \phi(s))ds, \int_{t_2}^{t} f(s, \psi(s))ds) \\ &\leq \int_{t_2}^{t_2+\eta} D^*(f(s, \phi(s)), f(s, \psi(s)))ds \\ &\leq \int_{t_2}^{t_2+\eta} kD^*(\phi(s), \psi(s))ds \leq k\eta H(\phi, \psi). \end{aligned}$$

for all $\phi, \psi \in C(I, (\mathbb{E}^1)^n)$. Hence by Banach's contraction mapping theorem, G has a unique fixed point, which by Lemma 3.4.16 is the desired solution to the problem.

Express T as a union of a finite family of intervals I_k with the length of each interval less than η. The above analysis guarantees the existence of the unique solution in each of the interval I_k. Piecewise joining of these solutions together gives the existence of the unique solution in whole interval T. □

Remark 3.4.18. *Let $A : T \to (\mathbb{E}^1)^{n\times n}$, be continuous matrix valued functions and $B : T \to (\mathbb{E}^1)^n$ is continuous. Let $g : T \times (\mathbb{E}^1)^n \to (\mathbb{E}^1)^n$ is given by*

$$g(t, x(t)) = A(t)x(t) + B(t).$$

Let $A(t) := [a_{ij}(t)]$ and $[\underline{a_{ij}^{\alpha}}(t), \overline{a_{ij}^{\alpha}}(t)]$ be the α-cut of $a_{ij}(t)$. By the proof of Corollary 3.4.8, $|\underline{a_{ij}^{\alpha}}(t)|$ and $|\overline{a_{ij}^{\alpha}}(t)|$ will be bounded by a constant K free from α and t for all $1 \leq i, j \leq n$. Then it can be shown that

$$D^*(f(t, x), f(t, y)) \leq nKD^*(x, y).$$

Hence by Thoerem 3.4.17, the fuzzy initial value problem

$$\dot{x}(t) = g(t, x(t)), x(t_0) = x_0$$

has a unique solution on T.

3.5 Conclusion

In the chapter, we have studied the behavior of solutions of nonlinear ODEs with fuzzy initial conditions and fuzzy inputs. Further, for the linear systems with fuzzy initial conditions and fuzzy inputs, we have obtained a closed form formula for the solutions using the transition matrix for the fuzzified linear system. Also, the existence and uniqueness of solutions of a fuzzy initial value problem in $(\mathbb{E}^1)^n$ is established. The results of the chapter can be treated as the generalizations to some of the results in the literature, for example, Kaleva (1987), Seikkala (1987) and Xu et al. (2007). Clearly, the present investigation enriches our knowledge about the solutions of the systems considered in this chapter.

CHAPTER 4

Controllability of Fuzzy Dynamical Systems

4.1 Introduction

Controllability of the classical crisp systems, that is, systems governed by continuous time or discrete time differential equations, has been extensively explored in the literature. So far in the literature only for linear systems the controllability conditions and computation of controls are easy to determine. However, unfortunately, many of the real world systems are nonlinear in nature and in general, for nonlinear systems no such simple criterion for controllability or computation of steering controls exist. Fuzzy logic control techniques have been proved quite useful to provide a cost effective and reasonably accurate model that collectively represents nonlinearities and uncertainties in the underline system. Recently, in the literature there has been growing interests in tackling the real world control problems by using the fuzzy logic control techniques.

Starting from the seminal work by Zadeh on fuzzy sets (cf. Zadeh (1965)), the area of fuzzy systems has been enormously grown. Broadly fuzzy systems are categorized in to three main categories, namely, pure fuzzy systems, that is, systems governed by fuzzy differential equations (cf. Ding and Kandel (2000a,b), Dubey and George (2013a,c), Feng and Hu (2006), Kwun et al. (2009), Murty and Kumar (2008a), Phu and Dung (2011)) or fuzzy relational equations (cf. Cai and Tang (2000), Farinwata and Vachtsevanos (1993), Gupta et al. (1986)), T-S fuzzy systems (cf. Biglarbegian et al. (2012), Chen et al. (2012), Gassara et al. (2010)) and fuzzy logic systems using fuzzifiers and defuzzifiers. To the best of our knowledge about the literature, controllability of fuzzy dynamic systems has been investigated from the two different prospects, namely, controllability of fuzzy dynamic systems governed by fuzzy relational equations and controllability of fuzzy dynamic systems governed by fuzzy differential equations or T-S fuzzy systems. Fuzzy logic control techniques provide a general framework to deal with the controllability of nonlinear systems or systems with uncertainty. The basic idea in fuzzy logic control techniques is to combine the differential equation (plant model) with the expert knowledge in the form of if-then rules or sometimes in terms of fuzzily modelled parameters. An study of this approach of intelligent control in which low level linear model of the system is incorporated with the high level supervisory control in the form of if-them rules is mainly studied by Ding and Kandel (2000a,b). Moreover, the authors in Ding and Kandel (2000c); Ding et al. (2000) carried out an observability study (a concept dual of controllability) for the similar fuzzy dynamic systems. Other methods, in which if-then rules are used, are mainly depending on the fact that local dynamics of the system in different state space regions is represented by different linear models depending on if-then fuzzy rules. Then, the overall system model is represented by the fuzzy interpolation of the various local state-space models. Many researchers have studied many important properties namely stability (cf. Feng et al. (1997)), controllability (cf. Biglarbegian et al. (2012), Chen et al. (2009)) and reliable control design (cf. Chen and Liu (2004)) of fuzzy dynamic systems based on this approach. Although this method of fuzzy blending of various state-space linear models based on if-then rules is computationally efficient, however it has a disadvantage that the overall model itself can incorporate some undesirable errors due to imprecise if-then rules.

There are other studies proposed in the literature without any explicit supervisory if-then rules and the fuzziness in the system is incorporated intrinsically through various parameters, for example, initial condition, plant parameters, and control variables. Xu et al. (2007, 2010) have systematically studied the evolution of time-invariant differential dynamical systems with fuzzy initial condition and fuzzy plant parameters, respectively. Many authors, for instance, Feng and Hu (2006), Ding and Kandel (2000a,b), Dubey and George (2013a, 2012a) and Kwun et al. (2009, 2008) have studied the controllability of fuzzy differential dynamical systems with intrinsic fuzziness; also computation of fuzzy controls that steer the given initial state to a prescribed target state is also provided.

In our work, we will investigate controllability of the fuzzy dynamic systems from the aspects of fuzzy differential equations. We consider linear systems with fuzzy initial condition and fuzzy inputs and establish results on the controllability properties of the system. We will also introduce a concept of 'fuzzy-controllability' a concept weaker than controllability and provide a computational procedure for estimation of controllable initial fuzzy states (cf. Dubey and George (2012a)). Our results on controllability can be regarded as the extension of some of the results in Feng and Hu (2006) and Ding and Kandel (2000a). In Feng and Hu (2006), the authors assumed the initial condition to be in $\mathbb{R}^n$, whereas we establish our results by assuming the initial condition to be in $(\mathbb{E}^1)^n$, a much wider class than $\mathbb{R}^n$. Furthermore, we prove that the controllability of the pair (A^*, B^*) obtained by flip operations (refer to Section 4.2) on the matrix pair (A, B) is equivalent to the controllability of the pair (A, B) and the pair $(|A|, |B|)$, together. These results are discussed in Section 4.2. In Section 4.3, we have improved some of the results on controllability of fuzzy dynamical systems due to Ding and Kandel (2000a,b). Particulary, we have relaxed the invertibility assumption of input-to-state matrix $B(t)$ which is assumed in Ding and Kandel (2000a). Our results are established for a general input-to-state matrix $B(t)$. We conclude the chapter in Section 4.4.

4.2 Controllability: Levelwise Approach

Our aim in this section is to investigate the controllability of the time-invariant systems of the type

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(t_0) = X_0, t_1 \geq t \geq t_0 \geq 0, \end{cases} \tag{4.1}$$

where A and B are the $n \times n$, and $n \times m$ real matrices respectively. The initial state $X_0 \in (\mathbb{E}^1)^n$ and the input $u(t) \in (\mathbb{E}^1)^m$ for each $t \in [t_0, t_1]$ and $u(\cdot)$ is fuzzy-integrable function in $[t_0, t_1]$.

Let us recall the structure of the solutions of systems (4.1). By Theorem 3.2.1, the solution of system (4.1) is given by the following lemma.

Lemma 4.2.1. *For* $\alpha \in [0, 1]$, *let* $x_k^\alpha(t) = [\underline{x_k^\alpha}(t), \overline{x_k^\alpha}(t)]$ *be the* α*-cut of* $x_k(t)$ *for* $1 \leq k \leq n$ *and* $u_j^\alpha(t) = [\underline{u_j^\alpha}(t), \overline{u_j^\alpha}(t)]$ *be the* $\alpha-$*cut of* $u_j(t)$ *for* $1 \leq j \leq m$ *then the* *evolution of system* (4.1) *is described by the following* $2n-$*differential equations*:

$$\begin{cases} \dot{\underline{x_k^\alpha}}(t) = \min((Az+Bw)_k : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)], \\ \qquad\qquad\qquad w_j \in [\underline{u_j^\alpha}(t), \overline{u_j^\alpha}(t)]) \\ \dot{\overline{x_k^\alpha}}(t) = \max((Az+Bw)_k : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)], \\ \qquad\qquad\qquad w_j \in [\underline{u_j^\alpha}(t), \overline{u_j^\alpha}(t)]) \\ \underline{x_k^\alpha}(t_0) = \underline{X_{0k}^\alpha} \\ \overline{x_k^\alpha}(t_0) = \overline{X_{0k}^\alpha}, \end{cases} \tag{4.2}$$

where $1 \leq k \leq n$, *and* $(Az+Bw)_k = \Sigma_{i=1}^n a_{ki}z_i + \Sigma_{j=1}^m b_{kj}w_j$ *is the* k^{th} *row of* $Az+Bw$.

We now introduce the following new variables:

$$\underline{x^\alpha}(t) := [\underline{x_1^\alpha}(t), \underline{x_2^\alpha}(t), \ldots, \underline{x_n^\alpha}(t)]^T,$$

$$\overline{x^\alpha}(t) := [\overline{x_1^\alpha}(t), \overline{x_2^\alpha}(t), \ldots, \overline{x_n^\alpha}(t)]^T,$$

where $[\underline{x_k^\alpha}(t), \overline{x_k^\alpha}(t)]$ is the α-cut of $x_k(t)$ for $1 \leq k \leq n$. $\underline{u^\alpha}(t)$ and $\overline{u^\alpha}(t)$ are similarly defined. We denote $x_*^\alpha(t) := [(\underline{x^\alpha}(t))^T, (\overline{x^\alpha}(t))^T]^T := [\underline{x_1^\alpha}(t), \ldots, \underline{x_n^\alpha}(t), \overline{x_1^\alpha}(t), \ldots, \overline{x_n^\alpha}(t)]^T$ a column vector of size $2n$. $u_*^\alpha(t)$ is similarly defined.

By using these variables, equations (4.2) can be represented in a compact form as given below. For $\alpha \in [0, 1]$,

$$\dot{x_*^\alpha}(t) = A^* x_*^\alpha(t) + B^* u_*^\alpha(t), x_*^\alpha(t_0) = X_{0\,*}^\alpha,$$

in which A^* and B^* are defined as follows:

(i) If A has all its entries non-negative then $A^* = M$ and $B^* = N$, where

$$M = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \qquad N = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix},$$

i.e., M is a block diagonal matrix of size $2n \times 2n$ and N is a block diagonal matrix of size $2n \times 2m$. We denote $M = [m_{ij}], 1 \leq i, j \leq 2n$ and $N = [n_{ij}], 1 \leq$

$i \leq 2n, 1 \leq j \leq 2m$. Let the symbol "$m_{ij} \longleftrightarrow m_{kl}$" means that the entry in i^{th} row and j^{th} column of M is swapped by the entry in k^{th} row and l^{th} column of M, and vice versa. "$n_{ij} \longleftrightarrow n_{kl}$" is similarly defined.

(ii) If A has some of its entries negative then A^* is obtained by the following flip operations on the entries of M.
$m_{ij} \longleftrightarrow m_{i(j+n)}$ if $1 \leqslant j \leqslant n$ and $m_{ij} < 0$,
$m_{ij} \longleftrightarrow m_{i(j-n)}$ if $n < j \leqslant 2n$ and $m_{ij} < 0$.

(iii) If B has some of its entries negative then B^* is obtained by the following flip operations on the entries of N.
$n_{ij} \longleftrightarrow n_{i(j+m)}$ if $1 \leqslant j \leqslant m$ and $n_{ij} < 0$,
$n_{ij} \longleftrightarrow n_{i(j-m)}$ if $m < j \leqslant 2m$ and $n_{ij} < 0$.

(iv) If $u(t) \in \mathbb{R}^m$ is a crisp vector instead of being a vector of fuzzy numbers, then B^* can be taken as N and in this case $u^\alpha_*(t) = [u^T(t), u^T(t)]^T$.

Remark 4.2.2. *It can be easily shown that for any matrix $A \in \mathbb{R}^{m\times n}$, we have*

$$(A^*)^T = (A^T)^*,$$

where A^T denotes the transpose of A.

The following examples illustrates the above defined flip operations.

Example 4.2.3. *Let*

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}, M = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 2 & -1. \end{bmatrix}.$$

Then A^* is given by

$$A^* = \begin{bmatrix} 0 & 2 & -1 & 0 \\ 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 \\ 0 & -1 & 2 & 0 \end{bmatrix}.$$

In A^*, the negative entries m_{11}, m_{22}, m_{33}, m_{44} of the matrix M are flipped by m_{13}, m_{24}, m_{31} and m_{42}, respectively.

We will now define the controllability for the fuzzy dynamical system (4.1).

Definition 4.2.4. *(Controllability) The system* (4.1) *with fuzzy initial condition $x(t_0) = X_0 \in (\mathbb{E}^1)^n$ is said to be controllable to a fuzzy-state $X_1 \in (\mathbb{E}^1)^n$ at $t_1(> t_0)$ if there exists a fuzzy-integrable control $u(t) \in (\mathbb{E}^1)^m$ for $t \in [t_0, t_1]$ such that the solution of system* (4.1) *with this control satisfies $x(t_1) = X_1$.*

We will now give sufficient conditions for the controllability of fuzzy dynamical system (4.1). Note that if the pair (A^*, B^*) is controllable then a control $u(\cdot)$ which steers the system $\dot{x}(t) = A^*x(t) + B^*u(t)$ from an initial state x_0 in $\mathbb{R}^{2n}$ to a desired state x_1 in $\mathbb{R}^{2n}$ during time interval $[t_0, t_1]$ is given by

$$\begin{aligned} u(t) &\triangleq \eta(t, t_0, t_1, x_0, x_1) \\ &:= B^{*T}\Phi^{*T}(t_0, t)W^{*-1}(t_0, t_1)[\Phi^*(t_0, t_1)x_1 - x_0], \end{aligned}$$

where $\Phi^*(t, \tau)$ denotes the transition matrix for the system $\dot{x}(t) = A^*x(t)$ and $W^*(t_0, t_1) := \int_{t_0}^{t1} \Phi^*(t_0, \tau)B^*B^{*T}\Phi^{*T}(t_0, \tau)d\tau$ is the controllability Grammian for the system $\dot{x}(t) = A^*x(t) + B^*u(t)$.

Theorem 4.2.5. *The system* (4.1) *with fuzzy initial condition $X_0 \in (\mathbb{E}^1)^n$ is controllable to $X_1 \in (\mathbb{E}^1)^n$ during time interval $[t_0, t_1]$ if*

(i) The Pair (A^*, B^*) is controllable.

(ii) The function $u(\cdot)$, characterized by $[u(t)]_\alpha = [\underline{u^\alpha}(t), \overline{u^\alpha}(t)]$, where $\underline{u^\alpha}(t), \overline{u^\alpha}(t)$ are defined by $[(\underline{u^\alpha}(t))^T, (\overline{u^\alpha}(t))^T]^T := \eta(t, t_0, t_1, X^\alpha_{0\,*}, X^\alpha_{1\,*})$, belongs to $(\mathbb{E}^1)^m$.

Proof. Since the evolution of the system (4.1) is given by the following levelwise set of equations:

$$\dot{x^\alpha_*}(t) = A^*x^\alpha_*(t) + B^*u^\alpha_*(t), \alpha \in [0, 1]. \tag{4.3}$$

Using condition (i), it follows that for each $\alpha \in [0, 1]$, there exists a control $\tilde{u}^\alpha_*(t) := \eta(t, t_0, t_1, X^\alpha_{0\,*}, X^\alpha_{1\,*})$ with which the solution of system (4.3) with initial crisp state

$x_*^\alpha(t_0) = X_{0\,*}^\alpha$ satisfies $x_*^\alpha(t_1) = X_{1\,*}^\alpha$. Condition (ii) now implies that there exists a function $\tilde{u}(\cdot)$ such that $\tilde{u}(t) \in (\mathbb{E}^1)^m$ for each $t \in [t_0, t_1]$ and $[(\underline{\tilde{u}^\alpha}(t))^T, (\overline{\tilde{u}^\alpha}(t))^T]^T = \eta(t, t_0, t_1, X_{0\,*}^\alpha, X_{1\,*}^\alpha)$. Since $\tilde{u}_*^\alpha(t)$ is integrable in $[t_0, t_1]$, therefore $\int_{t_0}^{t_1} \underline{\tilde{u}^\alpha}(t)$ and $\int_{t_0}^{t_1} \overline{\tilde{u}^\alpha}(t)$ are well defined, which implies that $\tilde{u}(t)$ is fuzzy-integrable in $[t_0, t_1]$. Hence $\tilde{u}(t)$ is a fuzzy-controller with which the solution of system (4.1) with fuzzy initial condition $x(t_0) = X_0$ satisfies

$$x_*^\alpha(t_1) = X_{1*}^\alpha \text{ for all } \alpha \in [0, 1].$$

From the above equation it follows that, for all $\alpha \in [0, 1]$,

$$\underline{x^\alpha}(t_1) = \underline{X_1^\alpha} \text{ and } \overline{x^\alpha}(t_1) = \overline{X_1^\alpha}.$$

From the last equation it follows that $x(t_1) = X_1$. Hence system (4.1) with initial condition $X_0 \in (\mathbb{E}^1)^n$ is controllable to the fuzzy state $X_1 \in (\mathbb{E}^1)^n$ during time interval $[t_0, t_1]$. □

Remark 4.2.6. *Note that in the proof of Theorem 4.2.5, we require pair (A^*, B^*) to be controllable only on a proper subset of $\mathbb{R}^{2n}$ therefore we can replace the condition (i) of Theorem 4.2.5 by a weaker condition. Let S^{2n} be a subset of $\mathbb{R}^{2n}$ such that $S^{2n} = \{x : x \in \mathbb{R}^{2n}, x(i) \leq x(i+n), 1 \leq i \leq n\}$. The condition (i) of Theorem 4.2.5 can be replaced by controllability of the pair (A^*, B^*) on S^{2n} and it is also the necessary condition for system* (4.1) *to be controllable, as shown in following theorem.*

Theorem 4.2.7. *If the system* (4.1) *is controllable then the pair (A^*, B^*) is controllable on S^{2n}.*

Proof. Let $x_0, x_1 \in S^{2n}$ be any two crisp states. Get two fuzzy-states X_0 and X_1 in $(\mathbb{E}^1)^n$ such that $x_0 = X_{0\,*}^\alpha$ and $x_1 = X_{1\,*}^\alpha$ for some $\alpha \in [0, 1]$. This is always possible because of the structure of S^{2n}. Since the system (4.1) is controllable, therefore it follows that there exists a fuzzy-controller $\tilde{u}(\cdot)$ with which the solution of (4.1) with initial condition $x(t_0) = X_0$ satisfies $x(t_1) = X_1$. Therefore, the control $u_*^\alpha(t) := \tilde{u}_*^\alpha(t)$ with values in $\mathbb{R}^{2m}$, will steer the levelwise decomposed system $\dot{x}_*^\alpha(t) = A^* x_*^\alpha(t) + B^* u_*^\alpha(t)$ from x_0 to x_1 during time interval $[t_0, t_1]$. Hence the pair (A^*, B^*) is controllable on S^{2n}. □

Since controllability of pair (A^*, B^*) is required in establishing the controllability of system (4.1). In general, checking the controllability conditions for the pair (A^*, B^*) is computationally inefficient due to the fact that the sizes of A^* and B^* are twice that of the original matrices A and B, respectively. However, alternatively, the controllability of the pair (A^*, B^*) can be checked in an efficient way as expressed by the following result. Let $|A|$ denotes the matrix of the size as that of A and whose entries are the absolute values of the corresponding entries in A. $|B|$ is similarly defined.

Theorem 4.2.8. *Pair (A^*, B^*) is controllable if and only if the pair (A, B) and the pair $(|A|, |B|)$ are both controllable.*

Proof. Assume that pair (A^*, B^*) is controllable. We want to show that (A, B) and $(|A|, |B|)$ are also controllable. We will prove it by the method of contradiction. Suppose first that the pair (A, B) is not controllable, then by PBH test of controllability, there exists a non-zero eigenvector v and an eigenvalue λ of A^T such that

$$A^T v = \lambda v \text{ and } B^T v = 0. \tag{4.4}$$

Define a vector $w = [v, v]^T$, then from Eq. (4.4) we have

$$(A^T)^* w = \lambda w \text{ and } (B^T)^* w = 0. \tag{4.5}$$

Since $(A^T)^* = (A^*)^T$ and $(B^T)^* = (B^*)^T$, Eq. (4.5) implies

$$A^{*T} w = \lambda w \text{ and } B^{*T} w = 0. \tag{4.6}$$

By PBH test, Eq. (4.6) implies that the pair (A^*, B^*) is not controllable contrary to the assumption. Similarly, if the pair $(|A|, |B|)$ is not controllable then there exists a non-zero vector $v \in \mathbb{R}^n$ such that

$$|A|^T v = \lambda v \text{ and } |B|^T v = 0. \tag{4.7}$$

Now, by taking $w = [v, -v]^T$, (4.6) follows from (4.7), which is again a contradiction.

Conversely, assume that (A, B) and $(|A|, |B|)$ are controllable, we want to show that the pair (A^*, B^*) is controllable. Suppose (A^*, B^*) is not controllable, then there exists a non-zero eigenvector vector $x = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{2n})$ and an eigenvalue λ of A^{*T} such that

$$A^{*T}x = \lambda x \text{ and } B^{*T}x = 0. \tag{4.8}$$

Now define a vector $v = (v_1, v_2, \ldots, v_n)$ such that $v_i = x_i + x_{n+i}$ for each $i = 1, 2, \ldots, n$. Then, from (4.8) it follows that

$$A^T v = \lambda v \text{ and } B^T v = 0.$$

The last equation is contrary to the fact that pair (A, B) is controllable. Next choose $v = (v_1, v_2, \ldots, v_n)$ such that $v_i = x_i - x_{n+i}$ for each $i = 1, 2, \ldots, n$. Then, from Eq. (4.8) we have

$$|A|^T v = \lambda v \text{ and } |B|^T v = 0.$$

The last equation shows that the pair $(|A|, |B|)$ is also not controllable, a contradiction. Hence the lemma. □

Remark 4.2.9. *Following closely the proof given above, it can also be shown that pair* $(|A^*|, |B^*|)$ *is controllable if and only if the pair* (A, B) *and the pair* $(|A|, |B|)$ *are both controllable.*

Using Theorem 4.2.8 and the remark thereof, the following corollary is obvious.

Corollary 4.2.10. *Pair* $(|A^*|, |B^*|)$ *is controllable if and only if the pair* (A^*, B^*) *is controllable.*

It is worthwhile to note that the condition (ii) of Theorem 4.2.5 inherently states that the controllability of system (4.1) not only depends on the matrices A and B but also on the initial and final fuzzy-states, whereas crisp-controllability of system (4.1) depends only on matrices A and B. Therefore, given any arbitrary initial state $X_0 \in (\mathbb{E}^1)^n$ it may not be possible to control the system to an arbitrary state $X_1 \in (\mathbb{E}^1)^n$. However, if the initial state is crisp, that is, $X_0 \in \mathbb{R}^n$ then the set of all reachable fuzzy states from X_0 can be characterized more precisely by using a result due to Feng and Hu (2006)[Theorem 3.4]. Thus, we have the following theorem.

Theorem 4.2.11. *The fuzzy control system* (4.1) *with the arbitrary initial condition* $x_0 \in \mathbb{R}^n$ *can be steered to any fuzzy state in the admissible controllable state subset* $(\mathbb{E}_0^1)^n$ *of* $(\mathbb{E}^1)^n$ *if and only if the pair* (A^*, B^*) *is controllable. And the admissible controllable state subset* $(\mathbb{E}_0^1)^n$ *of* $(\mathbb{E}^1)^n$ *is given by*:

$$\begin{aligned}(\mathbb{E}_0^1)^n = \{V \in (\mathbb{E}^1)^n \,|\, \overline{V}^1 - \underline{V}^1 \in & \bigcap_{t_0 \le t \le t_1} (\Psi(t))^{-1}\mathbb{R}_+^m \text{ and} \\ & \frac{d}{d\alpha}\begin{pmatrix} \underline{V}^\alpha \\ -\overline{V}^\alpha \end{pmatrix} \in \bigcap_{t_0 \le t \le t_1} (\Psi^*(t))^{-1}\mathbb{R}_+^{2m}, \\ & \alpha \in (0,1]\},\end{aligned}$$

where $\Psi(t)$, $\Psi^*(t)$ *are defined as follows*:

$$\Psi(t) = |B|^T \Phi_{|A|}^T(t_1, t) W_1^{-1}(t_1, t_0),$$

in which $\Phi_{|A|}(t,s)$ *is the transition matrix for the system* $\dot{x} = |A|x$ *and* $W_1(t_1, t_0)$ *is defined by*

$$W_1(t_1, t_0) = \int_{t_0}^{t_1} \Phi_{|A|}(t_1, s)|B||B|^T \Phi_{|A|}^T(t_1, s) ds.$$

$$\Psi^*(t) = |B^*|^T \Phi_{|A^*|}^T(t_1, t) W_2^{-1}(t_1, t_0),$$

where $\Phi_{|A^*|}(t,s)$ *is the transition matrix for the system* $\dot{x} = |A^*|x$ *and* $W_2(t_1, t_0)$ *is defined by*

$$W_2(t_1, t_0) = \int_{t_0}^{t_1} \Phi_{|A^*|}(t_1, s)|B^*||B^*|^T \Phi_{|A^*|}^T(t_1, s) ds.$$

Proof. By Corollary 4.2.10, controllability of the pair (A^*, B^*) is equivalent to the controllability of pair $(|A^*|, |B^*|)$. Now the proof follows along the similar lines of the proof of Theorem 3.4 of Feng and Hu (2006). □

We will now provide a closed form formula for the steering fuzzy control that can be applied to the systems of type (4.1) with the matrices A, B having non-negative entries. For a matrix A, $A \ge 0$, we mean that all the entries of A are non-negative. When $A \ge 0$ and $B \ge 0$, we have the following result.

Theorem 4.2.12. *Let* $A, B \ge 0$ *in system* (4.1) *and the controllability Grammian*

$W(t_0,t_1) := \int_{t_0}^{t_1} \Phi(t_0,t)BB^T\Phi^T(t_0,t)$ *for the system* (4.1) *is non singular then a fuzzy-controller* $u(\cdot)$*, which steers an initial fuzzy-state* $X_0 \in (\mathbb{E}^1)^n$ *to a desired fuzzy-state* $X_1 \in (\mathbb{E}^1)^n$ *during time interval* $[t_0,t_1]$*, is given by*

$$[u(t)]_\alpha = [B^T\Phi^T(t_0,t)W^{-1}(t_0,t_1)(\Phi(t_0,t_1)\underline{\widetilde{X_1}}^\alpha - \overline{X_0^\alpha}), \\ B^T\Phi^T(t_0,t)W^{-1}(t_0,t_1)(\Phi(t_0,t_1)\overline{\widetilde{X_1}}^\alpha - \underline{X_0^\alpha})] \tag{4.9}$$

provided $[u(t)]_\alpha$ *defines a fuzzy number in* $(\mathbb{E}^1)^m$ *and* $[\underline{\widetilde{X_1}}^\alpha, \overline{\widetilde{X_1}}^\alpha = [\underline{X_1^\alpha} + \Phi(t_1,t_0)(\overline{X_0^\alpha} - \underline{X_0^\alpha}), \overline{X_1^\alpha} - \Phi(t_1,t_0)(\overline{X_0^\alpha} - \underline{X_0^\alpha})]$.

Proof. Under the condition $A, B \geq 0$, the evolution of the system (4.1) is given by the following set of levelwise decomposed linear differential equations (see Lemma 4.2.1).

$$\begin{cases} \dot{\underline{x^\alpha}}(t) = A\underline{x^\alpha}(t) + B\underline{u^\alpha}(t) \\ \dot{\overline{x^\alpha}}(t) = A\overline{x^\alpha}(t) + B\overline{u^\alpha}(t) \\ \underline{x^\alpha}(t_0) = \underline{X_0^\alpha} \\ \overline{x^\alpha}(t_0) = \overline{X_0^\alpha}, \end{cases} \tag{4.10}$$

where $\alpha \in [0,1]$. From (4.9), $\underline{u^\alpha}(t)$ and $\overline{u^\alpha}(t)$ are obtained as below.

$$\underline{u^\alpha}(t) = B^T\Phi^T(t_0,t)W^{-1}(t_0,t_1)(\Phi(t_0,t_1)\underline{\widetilde{X_1}}^\alpha - \overline{X_0^\alpha}),$$

$$\overline{u^\alpha}(t) = B^T\Phi^T(t_0,t)W^{-1}(t_0,t_1)(\Phi(t_0,t_1)\overline{\widetilde{X_1}}^\alpha - \underline{X_0^\alpha}).$$

The solution of system (4.10) is given by following two equations:

$$\underline{x^\alpha}(t) = \Phi(t,t_0)\underline{X_0^\alpha} + \int_{t_0}^t \Phi(t,\tau)B\underline{u^\alpha}(\tau)d\tau. \tag{4.11}$$

$$\overline{x^\alpha}(t) = \Phi(t,t_0)\overline{X_0^\alpha} + \int_{t_0}^t \Phi(t,\tau)B\overline{u^\alpha}(\tau)d\tau. \tag{4.12}$$

From Eq. (4.11) we have,

$$\begin{aligned}
\underline{x^{\alpha}}(t_1) &= \Phi(t_1,t_0)\underline{X_0^{\alpha}} + \int_{t_0}^{t_1} \Phi(t_1,\tau)B\underline{u^{\alpha}}(\tau)d\tau, \\
&= \Phi(t_1,t_0)\underline{X_0^{\alpha}} + \int_{t_0}^{t_1} \Phi(t_1,\tau)BB^T\Phi^T(t_0,\tau) \\
&\qquad W^{-1}(t_0,t_1)(\Phi(t_0,t_1)\underline{\widetilde{X_1}^{\alpha}} - \overline{X_0^{\alpha}})d\tau, \\
&= \Phi(t_1,t_0)\underline{X_0^{\alpha}} + \\
&\qquad \Phi(t_1,t_0)WW^{-1}(\Phi(t_0,t_1)\underline{\widetilde{X_1}^{\alpha}} - \overline{X_0^{\alpha}}), \\
&= \underline{\widetilde{X_1}^{\alpha}} - \Phi(t_1,t_0)(\overline{X_0^{\alpha}} - \underline{X_0^{\alpha}}) = \underline{X_1^{\alpha}}. \qquad (4.13)
\end{aligned}$$

Similarly from Eq. (4.12), we can show that

$$\overline{x^{\alpha}}(t_1) = \overline{\widetilde{X_1}^{\alpha}} + \Phi(t_1,t_0)(\overline{X_0^{\alpha}} - \underline{X_0^{\alpha}}) = \overline{X_1^{\alpha}}. \qquad (4.14)$$

Equations (4.13) and (4.14) together imply that $x(t_1) = X_1$. Hence system (4.1) with the control $u(\cdot)$ given in Eq. (4.9) steers X_0 to X_1 during time interval $[t_0, t_1]$. □

Remark 4.2.13. *From Theorem 4.2.5 and Theorem 4.2.12, it is evident that not all states in $(\mathbb{E}^1)^n$ are controllable from an initial state in $(\mathbb{E}^1)^n$ even if the crisp system is controllable, that is, pair (A^*, B^*) is controllable. This is so because for certain target states $X_1 \in (\mathbb{E}^1)^n$ the controller $u(t)$ (as in condition (ii) of 4.2.5) steering the initial state X_0 to X_1 during $[t_0, t_1]$ need not necessarily belong to $(\mathbb{E}^1)^m$. It is easy to construct examples of the systems with initial fuzzy state X_0 at time t_0 and target fuzzy state X_1 at time t_1, for which the control $u(t)$ fails to be $(\mathbb{E}^1)^m$. Conclusively, complete controllability for the fuzzy systems, in general, can not be achieved. Also, against the intuition, the controllability of fuzzy system* (4.1) *is stronger than that of crisp system. that is, system* (4.1) *with crisp initial conditions ($i.e., X_0 \in \mathbb{R}^n$) and crisp inputs ($i.e., u(t) \in \mathbb{R}^m$).*

We shall give an example to show that not all fuzzy states are reachable from a given initial fuzzy state in $(\mathbb{E}^1)^n$.

Example 4.2.14. *Let*

$$\dot{x}(t) = x(t) + 2u(t)$$

and $x(0) = X_0$ and $x(1) = X_1$, where X_0 is the initial fuzzy state and X_1 is the desired fuzzy state in $\mathbb{E}^1$. X_0 and X_1 are defined as follows:

$$X_0(s) = \begin{cases} \frac{s}{2} + 1 & s \in [-2, 0] \\ -\frac{s}{2} + 1 & s \in [0, 2] \end{cases},$$

$$X_1(s) = \begin{cases} s & s \in [0, 1] \\ 2 - s & s \in [1, 2]. \end{cases}$$

The α-cut of X_0 is given by

$$[X_0]_\alpha = [2(\alpha - 1), 2(1 - \alpha)]$$

and the α cut of X_1 given by

$$[X_1]_\alpha = [\alpha, 2 - \alpha].$$

According to the Theorem 4.2.12 the α-level sets $[u(t)]_\alpha$ of the control $[u(t)]$ is given by

$$[u(t)]_\alpha = \Big[\frac{e^{-t}}{(1 - e^2)}(e^{-1}\underline{\widetilde{X_1}}^\alpha - \overline{X_0^\alpha}), \frac{e^{-t}}{(1 - e^2)}(e^{-1}\overline{\widetilde{X_1}}^\alpha - \underline{X_0^\alpha})\Big], \tag{4.15}$$

where

$$\begin{aligned} \underline{\widetilde{X_1}}^\alpha &= \underline{X_1}^\alpha + e(\overline{X_0}^\alpha - \underline{X_0}^\alpha) \\ &= \alpha + 4e(1 - \alpha), \end{aligned}$$

and

$$\begin{aligned} \overline{\widetilde{X_1}}^\alpha &= \overline{X_1}^\alpha - e(\overline{X_0}^\alpha - \underline{X_0}^\alpha) \\ &= (2 - \alpha) - 4e(1 - \alpha). \end{aligned}$$

Now from Eq. (4.15)*, we have*

$$[u(t)]_\alpha = \Big[\frac{e^{-t}}{(1 - e^2)}(e^{-1}\alpha + 2(1 - \alpha)), \frac{e^{-t}}{(1 - e^2)}(e^{-1}(2 - \alpha) - 2(1 - \alpha))\Big].$$

We will now show that the control $u(t)$ characterized by $[u(t)]_\alpha$ does not belong to

$\mathbb{E}^1$. *This can be proved by simply showing that* $\underline{u^\alpha}(t) > \overline{u^\alpha}(t)$. *Since* $\frac{e^{-t}}{(1-e^2)}$ > *0 and* $e^{-1}\alpha + 2(1-\alpha) > e^{-1}(2-\alpha) - 2(1-\alpha)$ *for all* $\alpha \in [0,1)$. *Thus, it follows that* $\underline{u^\alpha}(t) > \overline{u^\alpha}(t)$. *Hence* $u(t) \notin \mathbb{E}^1$.

We will now provide examples to demonstrate controllability of systems of the type (4.1). Examples 4.2.15, 4.2.16 apply to Theorem 4.2.12 and Theorem 4.2.5, respectively.

Example 4.2.15. *Let*

$$\dot{x}(t) = x(t) + 2u(t)$$

and $x(0) = X_0$ *and* $x(1) = X_1$*, where* X_0 *and* X_1 *are in* $\mathbb{E}^1$ *and are defined as follows*:

$$X_0(s) = \begin{cases} e^{1-\frac{1}{1-4s^2}} & |s| \le \frac{1}{2} \\ 0 & |s| \ge \frac{1}{2} \end{cases},$$

$$X_1(s) = \begin{cases} e^{1-\frac{4}{4-s^2}} & |s| \le 2 \\ 0 & |s| \ge 2. \end{cases}$$

In the setting of above example, we have $\Phi(t,\tau) = e^{t-\tau}$ *and* $W(0,1) = 2(1-e^2)$*. Using equation* (4.9) *the fuzzy-controller, which steers the initial fuzzy state* X_0 *to target fuzzy state* X_1 *during time-interval* $[0,1]$*, is given by*

$$u(t) = \frac{e^{-t}}{(1-e^2)}[e^{-1}\widetilde{X_1} - X_0],$$

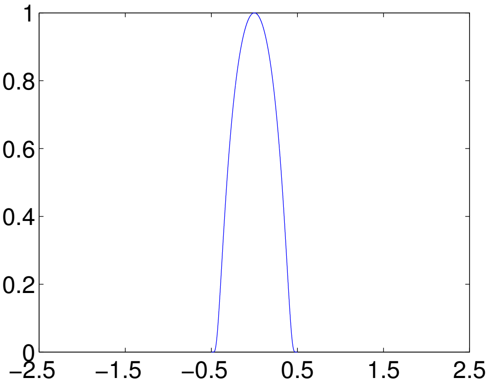
where the fuzzy number $\widetilde{X_1}$ *is defined as follows*:

$$\forall \alpha \in (0,1], [\widetilde{X_1}]_\alpha = [\underline{X_1^\alpha} + e(\overline{X_0^\alpha} - \underline{X_0^\alpha}), \overline{X_1^\alpha} - e(\overline{X_0^\alpha} - \underline{X_0^\alpha}).$$

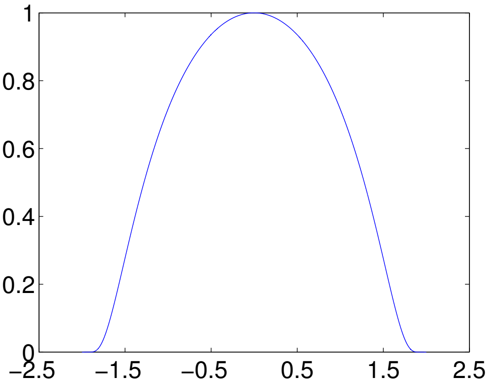
The propagated state at time $t = 1$ *(Figure 4.1d) coincides with the desired target state (Figure 4.1b). In Figure 4.2, lower and upper cuts of the control and system-states are plotted corresponding to* $\alpha = .5$*. It can be seen in the Figure (4.2b) that* $[X_0]_{\cdot 5}$ *is steered to* $[X_1]_{\cdot 5}$ *during time-interval* $[0,1]$.

Example 4.2.16. *Let*

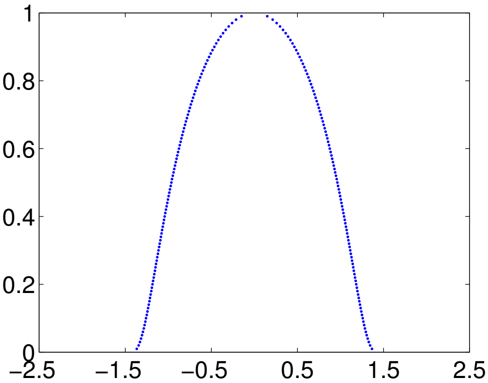
$$\dot{x}(t) = -x(t) - 2u(t)$$



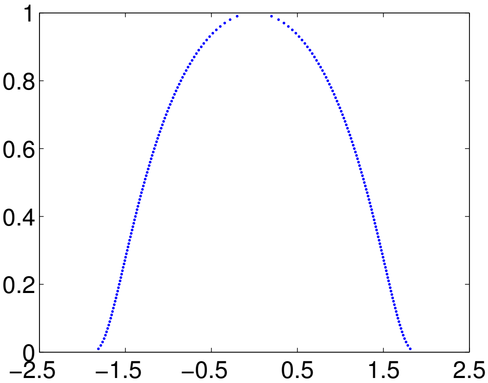
(a) Initial state at $t = 0$



(b) Desired target state at $t = 1$

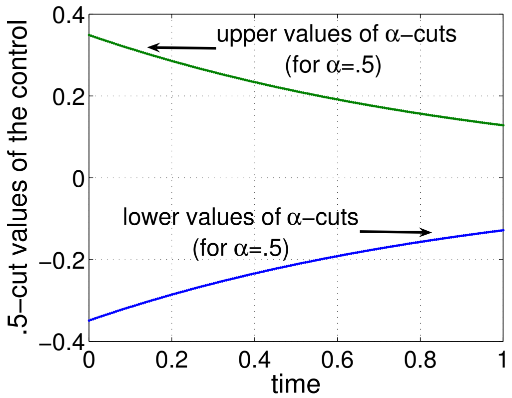


(c) Propagated system-state at $t = 3/4$

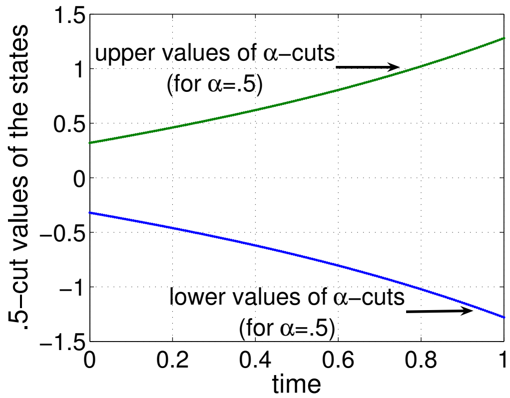


(d) Propagated system-state at $t = 1$

Figure 4.1: Initial, target and propagated states of the system



(a) Control plot for $\alpha = \cdot 5$



(b) State plot for $\alpha = \cdot 5$

Figure 4.2: Control and state plots for $\alpha = \cdot 5$ during $[0, 1]$

and $x(0) = X_0$ and $x(1) = X_1$, where X_0 and X_1 are in $\mathbb{E}^1$ and are defined as follows:

$$X_0(s) = \begin{cases} 2s & 0 \leq s \leq \frac{1}{2} \\ 2 - 2s & \frac{1}{2} \leq s \leq 1 \end{cases},$$

$$X_1(s) = \begin{cases} \frac{s}{4} & 0 \leq s \leq 4 \\ 2 - \frac{s}{4} & 4 \leq s \leq 8. \end{cases}$$

In this case, the evolution of system is given by the following level-wise equations:

$$\begin{pmatrix} \dot{\underline{x^{\alpha}}}(t) \\ \dot{\overline{x^{\alpha}}}(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \underline{x^{\alpha}}(t) \\ \underline{x^{\alpha}}(t) \end{pmatrix} +$$

$$\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} \underline{u^{\alpha}}(t) \\ \overline{u^{\alpha}}(t) \end{pmatrix}.$$

Using Theorem 4.2.5, the fuzzy-controller $u(\cdot)$ which steers X_0 to X_1 during time-interval $[0, 1]$, is given by the following α-cut representation:

$$\begin{aligned} [u(t)]_{\alpha} =& [-1.399e^{t} + e^{-t}(1.1238\alpha - 1.349), \\ & -1.399e^{t} + e^{-t}(-1.1238\alpha + 1.349)]. \end{aligned} \tag{4.16}$$

It is clear from Figure 4.3 that the initial fuzzy-state X_0 is steered to the desired target state X_1 during time-interval $[0, 1]$. In Figure 4.4, lower and upper α-cuts of the control and system-states are plotted corresponding to $\alpha = .5$. It can be easily seen in the Figure (4.4b) that $[X_0]_{.5}$ is steered to $[X_1]_{.5}$ during time-interval $[0, 1]$.

4.2.1 Fuzzy-controllability

Now we will introduce the concept of fuzzy-controllability for system (4.1), a concept weaker than controllability. Terdpravat (2004) in his MS thesis has introduced a similar concept called "controllability in fuzzy sense" and proved some of the results associated with it. However, there were some discrepancies in some of his results which we have rectified and that resulted in to the present investigation.

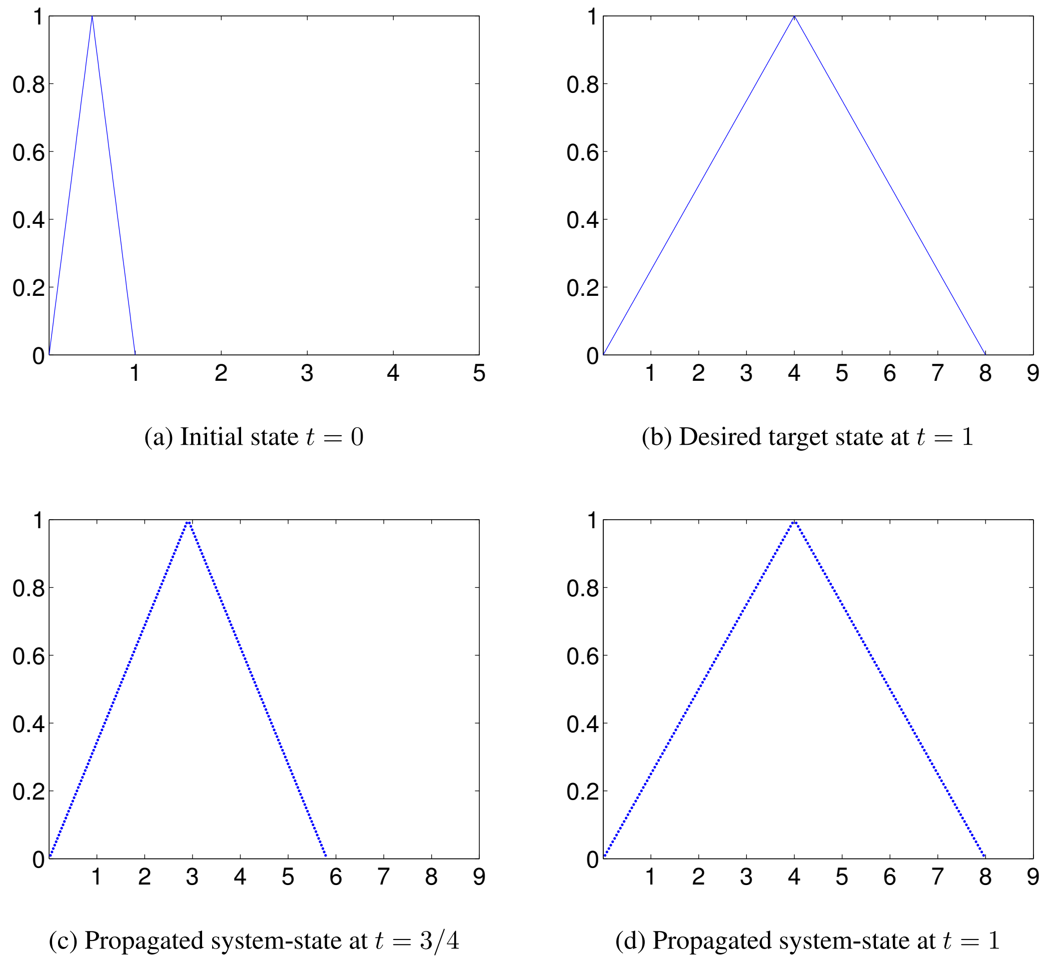


Figure 4.3: Initial, target and propagated states of the system

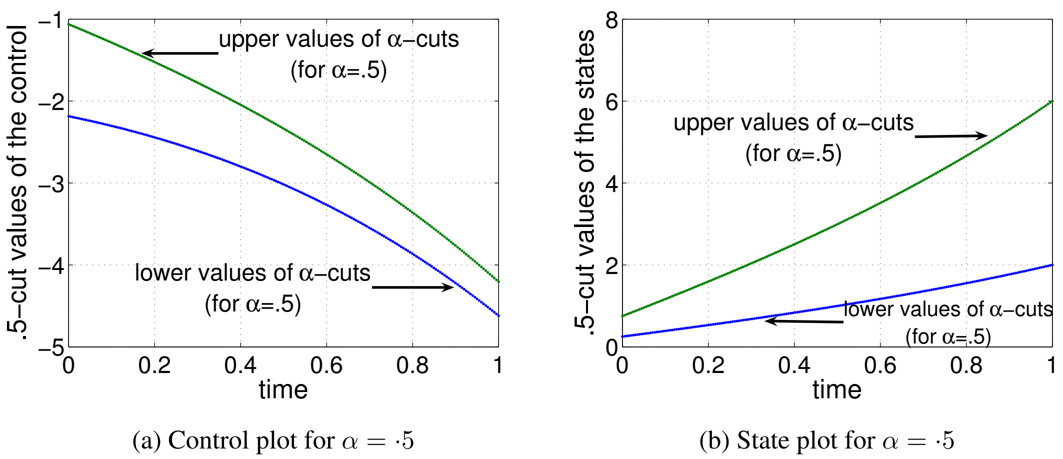


Figure 4.4: Control and state plots for $\alpha = \cdot 5$ during $[0, 1]$

In fuzzy-controllability one looks for a control such that the initial fuzzy state can be steered to within the desired target fuzzy state. We provide an estimate of initial fuzzy state that can be controlled to a predefined target fuzzy state, that is, given any two crisp states x_0, x_1 in $\mathbb{R}^n$, and a fuzzy state X_1 around x_1, we compute a fuzzy state X_0 around x_0 so that X_0 is fuzzy-controllable to X_1. In a special case, when the plant matrix has non-negative entries, the fuzzy state X_0 is fuzzy-controllable to X_1 with the crisp control that steers x_0 to x_1.

We will now define the fuzzy-controllability of system (4.1).

Definition 4.2.17. *System* (4.1) *with fuzzy initial condition* $x(t_0) = X_0 \in (\mathbb{E}^1)^n$ *is said to be fuzzy-controllable to a fuzzy state* $X_1 \in (\mathbb{E}^1)^n$ *at* $t_1 (> t_0)$ *if there exists a fuzzy-integrable control* $u(\cdot) \in (\mathbb{E}^1)^n$ *such that the solution of system* (4.1) *with this control satisfies* $x(t_1) \leqslant X_1$.

By a fuzzy state $X \in (\mathbb{E}^1)^n$ around $x \in \mathbb{R}^n$, we mean that $X_*^1 = [x, x]^T$, that is, $\underline{X^1} = \overline{X^1} = x$. If $X \in (\mathbb{E}^1)^n$ around $x \in \mathbb{R}^n$ with $X \neq x$, in the sense of natural imbedding of x in $(\mathbb{E}^1)^n$, then we call X as a nontrivial fuzzy state around x.

We will now show that given a target fuzzy state X_1 in $(\mathbb{E}^1)^n$ around x_1, we can always find a nontrivial fuzzy state X_0 in $(\mathbb{E}^1)^n$ around x_0 so that X_0 is fuzzy-controllable to X_1. The following theorem gives an estimate of initial fuzzy state X_0 with which the system (4.1) is fuzzy-controllable to the target fuzzy state X_1.

Theorem 4.2.18. *Let the system pair* (A^*, B^*) *be controllable and* x_0, $x_1 \in \mathbb{R}^n$. *Assume further that there exists a control* $u^*(\cdot) \in L^2([t_0, t_1]; \mathbb{R}^{2m})$ *with the following properties*:

(H1) The crisp system $\dot{x}(t) = A^* x(t) + B^* u(t)$ *with the control* $u^*(\cdot)$ *steers* $[x_0, x_0]^T$ *to* $[x_1, x_1]^T$ *during time interval* $[t_0, t_1]$.

(H2) For $1 \leq k \leq m$, $u_k^*(t) \leq u_{k+m}^*(t)$ *in which* $u_k^*(t)$ *is the* k^{th} *component of* $u^*(t)$.

Then given any nontrivial fuzzy state $X_1 \in (\mathbb{E}^1)^n$ *around* x_1, *there exist a nontrivial fuzzy state* $X_0 \in (\mathbb{E}^1)^n$ *around* x_0 *and a control* $u(\cdot) \in (\mathbb{E}^1)^m$ *such that the system* (4.1) *with fuzzy initial condition* X_0 *and the control* $u(\cdot)$ *is fuzzy-controllable to* X_1 *during time interval* $[t_0, t_1]$.

Proof. Using hypothesis (H2), define $u(t) = [u_1(t), u_2(t), \ldots, u_m(t)]^T \in \mathbb{E}^m$ for $t \in [t_0, t_1]$ in which each $u_k(t)$ for $1 \le k \le m$ is a rectangular fuzzy number on $\mathbb{R}$ defined by the following $\alpha-$cut representation. For $\alpha \in (0, 1]$,

$$[u_k(t)]_\alpha = [u_k^*(t), u_{k+m}^*(t)].$$

Clearly $u(\cdot)$ is fuzzy-integrable. Let $X_0 \in (\mathbb{E}^1)^n$ be a fuzzy state around x_0, which we will characterize at the end of proof. Assume without loss of generality that $[X_1]_\alpha = [x_1 - \boldsymbol{\gamma}^\alpha, x_1 + \boldsymbol{\gamma}^\alpha]$, where $\boldsymbol{\gamma}^\alpha = [\gamma_1^\alpha, \gamma_2^\alpha, \ldots, \gamma_n^\alpha]^T$ with $\gamma_k^\alpha \in \mathbb{R}^+$ for $1 \le k \le n$, $\alpha \in (0, 1]$, that is, $\alpha-$cut of the k^{th} component of X_1 is given by $[x_{1k} - \gamma_k^\alpha, x_{1k} + \gamma_k^\alpha]$. The evolution of system (4.1), with fuzzy initial condition X_0 and the control $u(\cdot)$ as defined above, is given by the following equations (see Lemma 4.2.1). For $\alpha \in (0, 1]$,

$$\begin{cases} \dot{x_*^\alpha}(t) = A^* x_*^\alpha(t) + B^* u_*^\alpha(t) \\ x_*^\alpha(t_0) = X_{0*}^{\alpha}, t \ge t_0, \end{cases} \tag{4.17}$$

The solution of system (4.17) at time t_1 is given by

$$x_*^\alpha(t_1) = \Phi^*(t_1, t_0) x_*^\alpha(t_0) + \int_{t_0}^{t_1} \Phi^*(t_1, \tau) B^* u^*(\tau) d\tau, \tag{4.18}$$

where $\Phi^*(t, \tau)$ denotes the transition matrix for system $\dot{x}(t) = A^* x(t)$. Denote $[x_1, x_1]^T$, a vector in $\mathbb{R}^{2n}$, by x_{*1} and similarly $[x_0, x_0]^T$ by x_{*0}. Setting $\alpha = 1$ in Eq. (4.18) and using hypothesis (H1), we have

$$x_{*1} = \Phi^*(t_1, t_0) x_*^0 + \int_{t_0}^{t_1} \Phi^*(t_1, \tau) B^* u^*(\tau) d\tau \tag{4.19}$$

Subtracting (4.19) from (4.18), we have

$$\|x_*^\alpha(t_1) - x_{*1}\| \le \|\Phi^*(t_1, t_0)\| \ \|x_*^\alpha(t_0) - x_{*0}\|, \tag{4.20}$$

where $\|\Phi^*(t_1, t_0)\| = \sup\limits_{x \ne 0} \left(\frac{\|\Phi^*(t_1, t_0) x\|}{\|x\|} \right)$ and $\|.\|$ denotes the 2-norm. From equation (4.20) it follows that

$$\|x_*^\alpha(t_1) - x_{*1}\|_\infty \le \sqrt{2n} \|\Phi^*(t_1, t_0)\| \, \|x_*^\alpha(t_0) - x_{*0}\|_\infty, \tag{4.21}$$

where $\|.\|_\infty$ denotes the infinity-norm. Let $\beta^\alpha = \frac{\gamma^\alpha_{min}}{\sqrt{(2n)}\,\|\Phi^*(t_1,t_0)\|}$, where $\boldsymbol{\gamma}^\alpha_{min} = \min[\gamma^\alpha_1, \gamma^\alpha_2, \ldots, \gamma^\alpha_n]$. Now define the fuzzy initial state $x(t_0) = X_0 \in (\mathbb{E}^1)^n$ by $[X_0]_\alpha = [x_0 - \boldsymbol{\beta}^\alpha, x_0 + \boldsymbol{\beta}^\alpha]$, where $\boldsymbol{\beta}^\alpha = [\beta^\alpha, \beta^\alpha, \ldots, \beta^\alpha]^T \in \mathbb{R}^n$, $\alpha \in (0, 1]$. With such a choice of fuzzy initial state X_0, Eq. (4.21) implies

$$\|x^\alpha_*(t_1) - x_{*1}\|_\infty \leq \boldsymbol{\gamma}^\alpha_{min}$$

which in turn shows that the propagated fuzzy state at time t_1 satisfies $x(t_1) \leq X_1$. Hence X_0 is the required initial fuzzy state with which system (4.1) is fuzzy-controllable to X_1 with the control $u(\cdot)$ during time interval $[t_0, t_1]$. □

Remark 4.2.19. *If the entries of the matrix A are non-negative, then the control $u(\cdot) \in (\mathbb{E}^1)^m$ can be taken as a crisp control $\tilde{u}(\cdot)$ with values in $\mathbb{R}^m$ that steers x_0 to x_1 during time interval $[t_0, t_1]$. That is, $u(\cdot) \in (\mathbb{E}^1)^m$ is obtained from $\tilde{u}(\cdot) \in \mathbb{R}^m$ via the natural embedding of $\mathbb{R}^m$ into $(\mathbb{E}^1)^m$. Thus, for $1 \leq k \leq n$, $u_k(t) \in \mathbb{E}$ is obtained as follows:*

$$u_k(t)(s) = \begin{cases} 1 & \text{if } s = \tilde{u}_k(t) \\ 0 & \text{if } s \neq \tilde{u}_k(t), \end{cases} \tag{4.22}$$

where $u_k(\cdot)$ and $\tilde{u}_k(\cdot)$ are the k^{th} component of $u(\cdot)$ and $\tilde{u}(\cdot)$, respectively.

Remark 4.2.20. *In the case, when A has all its entries nonnegative then the constant β^α can be taken as $\frac{\gamma^\alpha_{min}}{\sqrt{(n)}\,\|\Phi(t_1,t_0)\|}$ because of the block diagonal structure of A^*. Here $\Phi(t, \tau)$ denotes the transition matrix for the system $\dot{x}(t) = Ax(t)$*

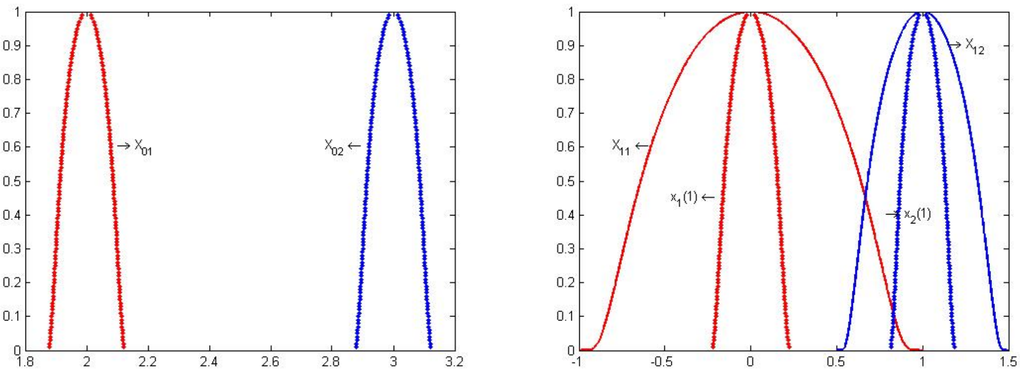
In the following example, we compute a controllable initial fuzzy state that can be controlled to a desired target fuzzy state.

Example 4.2.21. *Consider the following differential equation*

$$\begin{pmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{pmatrix} = \begin{pmatrix} \cdot 5 & \cdot 1 \\ \cdot 1 & \cdot 3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} u(t). \tag{4.23}$$

Let $x_0 = [2, 3]^T$, $x_1 = [0, 1]^T \in \mathbb{R}^2$ and $X_1 = [X_{11}, X_{12}]^T \in \mathbb{E}^2$, a fuzzy state around x_1, where X_{11}, X_{12} are given by the following membership functions:

$$X_{11}(s) = e^{1 - \frac{1}{1-s^2}}, \qquad X_{12}(s) = e^{1 - \frac{1}{1-4(s-1)^2}}.$$



(a) Computed initial fuzzy state X_0

(b) Target fuzzy state X_1 and the propagated fuzzy state $x(t)$ at $t = 1$

Figure 4.5: Initial fuzzy state X_0 (a) is fuzzy-controllable to a desired target fuzzy state X_1 (b)

Here the steering control $u(\cdot)$ is taken as minimum energy control (refer to Eq. (2.4) *of Chapter 2) that steers x_0 to x_1 during time interval* $[0, 1]$*(see Remark 4.2.19). In the setting of above example, we have* $\boldsymbol{\gamma}^\alpha = \left[\left(1 - \frac{1}{log\left(\frac{e}{\alpha}\right)}\right)^{\frac{1}{2}}, \frac{1}{2}\left(1 - \frac{1}{log\left(\frac{e}{\alpha}\right)}\right)^{\frac{1}{2}}\right]^T$. *Therefore,* $\beta^\alpha = \frac{\boldsymbol{\gamma}^\alpha_{min}}{\sqrt{(2)}\|e^A\|} = \frac{\gamma_2^\alpha}{3.6975}$. *The fuzzy number* $X_0 = [X_{01}, X_{02}]^T$ *will be given by the following α-cut representation*:

$$[X_{01}]_\alpha = [2 - \beta^\alpha, 2 + \beta^\alpha] \qquad [X_{02}]_\alpha = [3 - \beta^\alpha, 3 + \beta^\alpha].$$

It is clear from Figure 4.5 that system (4.23) *with fuzzy initial state X_0 and control $u(\cdot)$ is fuzzy-controllable to X_1 during time interval* $[0, 1]$.

4.3 Controllability: Differential Inclusion Approach

Ding and Kandel (2000a,b) studied the controllability of the following fuzzy dynamical control system (FDCS):

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)U(t) \\ x(0) = x_0, T_1 \geq t \geq 0, \end{cases} \tag{4.24}$$

where $x(t), U(t) \in \mathbb{E}^n$ and $x_0 \in \mathbb{R}^n$, $A(t), B(t)$ are $n \times n$ continuous real matrices in time interval $[0, T_1]$, where T_1 is finite time. In Ding and Kandel (2000a,b), the expressions for controls that steer the system (4.24) from an initial state $x_0 \in \mathbb{R}^n$ to a desired fuzzy state $X_1 \in \mathbb{E}^n$ are derived. However, the authors have assumed the invertibility of the matrix $B(t)$. In many practical situations, the control matrix $B(t)$ need not be invertible, more generally it need not even be a square matrix. This is due to the fact that number of control variables must be kept as minimum as possible, therefore we seek the control $u(t)$ in $\mathbb{E}^m$ where $m < n$. Thus, we study controllability of the following system in which $B(t)$ is $n \times m$ matrix:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)U(t) \\ x(0) = x_0 \in \mathbb{R}^n, T_1 \geq t \geq 0, \end{cases} \tag{4.25}$$

where $x(t) \in (\mathbb{E}^1)^n$ and $u(t) \in (\mathbb{E}^1)^m$. $A(t), B(t)$ are $n \times n$, $n \times m$ continuous real matrices in time interval $[0, T_1]$, respectively. Also, we obtain new formulae for the controls that steer the system (4.25) from an initial state $x_0 \in \mathbb{R}^n$ to a desired fuzzy state $X_1 \in (\mathbb{E}^1)^n$. Before establishing the controllability results for the system (4.25), we will briefly describe the solutions of the system (4.25). We will follow the approach due to Ding and Kandel (2000a) for the evolution of system (4.25). The evolution of system (4.25) is given by the following family of differential inclusions:

$$\begin{cases} \dot{x}(t) & \in A(t)x(t) + B(t)U^\alpha(t), t \in [0, T_1] \\ x(0) & = x_0, \end{cases} \tag{4.26}$$

where $U^\alpha(t) = U_1^\alpha(t) \times U_2^\alpha(t) \times \ldots \times U_m^\alpha(t)$ and for $1 \leq i \leq m$, $U_i^\alpha(t)$ be the α-level set of $U_i(t)$. Let X^α be the solution set to (4.26). It can be shown that X^α is nonempty, compact and convex in $C([0, T_1], \mathbb{R}^n)$ (for proof refer to Section 3 of Ding and Kandel (2000a)). Now from the Arzela-Ascoli theorem we have that $X^\alpha(t)$ is compact in $\mathbb{R}^n$. Furthermore, $X^\alpha(t)$ is convex in $\mathbb{R}^n$; this can be quickly seen as follows: Let $x_1^\alpha(\cdot), x_2^\alpha(\cdot) \in X^\alpha$. Then, there exist inputs $u_1^\alpha(t), u_2^\alpha(t) \in U^\alpha(t)$ for $t \in [0, T_1]$, such that

$$\dot{x}_1^\alpha(t) = A(t)x_1^\alpha(t) + B(t)u_1^\alpha(t), \tag{4.27}$$

$$\dot{x}_2^{\alpha}(t) = A(t)x_1^{\alpha}(t) + B(t)u_2^{\alpha}(t). \tag{4.28}$$

Using equations (4.27), (4.28) and given any $\beta \in [0,1]$, we have

$$\beta\dot{x}_1^{\alpha}(t)+(1-\beta)\dot{x}_2^{\alpha}(t) = A(t)(\beta x_1^{\alpha}(t)+(1-\beta)x_2^{\alpha}(t))+B(t)(\beta u_1^{\alpha}(t)+(1-\beta)u_2^{\alpha}(t)). \tag{4.29}$$

Since $U^{\alpha}(t)$ is convex, therefore $\beta u_1^{\alpha}(t) + (1-\beta)u_2^{\alpha}(t) \in U^{\alpha}(t)$ for all $t \in [0, T_1]$, hence it follows that $\beta x_1^{\alpha}(t) + (1-\beta)x_2^{\alpha}(t) \in X^{\alpha}(t)$ for all $t \in [0, T_1]$. Thus, we have shown that $X^{\alpha}(t) \in \mathbb{P}_k(\mathbb{R}^n)$.

Next, we want to show that as α varies in $[0,1]$ the family $X^{\alpha}(t)$ is a fuzzy set in $(\mathbb{E}^1)^n$. In order to do so we need to check the conditions of Theorem 2.2.10. We have already shown that $X^{\alpha}(t) \in \mathbb{P}_k(\mathbb{R}^n)$. Indeed, the remaining two conditions can be verified in the same manner as shown in Ding and Kandel (2000a). Therefore, we are skipping the details.

Thus, there exists $X(t) \in (\mathbb{E}^1)^n$ on $[0, T_1]$ such that $X^{\alpha}(t)$ is a solution set to differential inclusion (4.26). Hence, the system (4.25) is a fuzzy dynamical control system and the solution set to the equation (4.26) can be given by:

$$X^{\alpha}(t) = \Phi(t,0)x_0 + \int_0^t \Phi(t,\tau)B(\tau)U^{\alpha}(\tau)d\tau, \tag{4.30}$$

in which $\Phi(t,\tau)$ denotes the transition matrix corresponding to the homogeneous linear system

$$\dot{x}(t) = A(t)x(t).$$

Combining Eq. (4.30) for all $\alpha \in [0,1]$, the solution for the fuzzy dynamical control systems can be given by (cf. Ding and Kandel (2000a))

$$X(t) = \Phi(t,0)x_0 + \int_0^t \Phi(t,\tau)B(\tau)U(\tau)d\tau. \tag{4.31}$$

Now the controllability is concerned with the following problem: Given the system (4.25), for the initial state x_0, and a fuzzy state X_1 at time T_1, find a control $U(t)$, $t \in [0, T_1]$, that transfers x_0 (at 0) to X_1 (at T_1). The following definition of controllability is due to Ding and Kandel (2000a).

Definition 4.3.1. *(Controllability) The state* x_0 *of the system* (4.24) *is said to be controllable on time interval* $[0, T_1]$ *if some control* $U(\cdot)$ *over* $[0, T_1]$ *exists which transfers* x_0 *(at 0) to a fuzzy state at* T_1*. Otherwise the state* x_0 *is said to be uncontrollable on* $[0, T_1]$.

Throughout the section, let controllability Grammian $W(0, T_1)$ be given by

$$W(0, T_1) = \int_0^{T_1} \Phi(T_1, t)B(t)B^T(t)\Phi^T(T_1, t)dt.$$

For the controllability of the system (4.24) (when $B(t)$ is invertible), Ding and Kandel (2000a) obtained some sufficient conditions as given in following theorem.

Theorem 4.3.2 (Ding and Kandel (2000a))**.** *The system* (4.24) *is controllable over* $[0, T_1]$*, if* $W(0, T_1)$ *is non-singular and* $B(t)^{-1}$ *exists for all* t*. Furthermore, the control* $U(t)$ *which transfers the state of the system from* $x(0) = x_0 \in \mathbb{R}^n$ *to a fuzzy state* $x(T_1) = X_1 \in \mathbb{E}^n$ *can be chosen as*

$$U(t) = \frac{1}{T_1}B^{-1}\Phi^{-1}(T_1, t)X_1 - B^T(t)\Phi^T(T_1, t)W^{-1}(0, T_1)\Phi(T_1, 0)x_0.$$

Proof. (Ding and Kandel)

$$W(0, T_1) = \int_0^{T_1} \Phi(T_1, t)B(t)B^T(t)\Phi^T(T_1, t)dt. \tag{4.32}$$

Now post-multiply both sides of (4.32) by $W^{-1}(0, T_1)\Phi(T_1, 0)x_0$ to obtain

$$\Phi(T_1, 0)x_0 = \int_0^{T_1} \Phi(T_1, t)B(t)B^T(t)\Phi^T(T_1, t)W^{-1}(0, T_1)\Phi(T_1, 0)x_0 dt. \tag{4.33}$$

If $U(t)$ exists such that $U(t)$ transfer x_0 to X_1 during $[0, T_1]$, then from Eq. (4.31) we get

$$x(T_1) = X_1 = \Phi(T_1, 0)x_0 + \int_0^{T_1} \Phi(T_1, t)B(t)U(t)dt. \tag{4.34}$$

Also by using the fact that $X_1 = \frac{1}{T_1}\int_0^{T_1} X_1 dt$ we have the following relation

$$X_1 = \frac{1}{T_1}\int_0^{T_1} \Phi(T_1, t)B(t)B^{-1}(t)\Phi^{-1}(T_1, t)X_1 dt. \tag{4.35}$$

Substituting (4.33) and (4.35) in (4.34), we get,

$$\frac{1}{T_1}\int_0^{T_1}\Phi(T_1,t)B(t)B^{-1}(t)\Phi^{-1}(T_1,t)X_1 = \int_0^{T_1}\Phi(T_1,t)B(t)B^T(t)\Phi^T(T_1,t)W^{-1}(0,T_1)\Phi(T_1,0)x_0dt + \int_0^{T_1}\Phi(T_1,t)B(t)U(t)dt. \tag{4.36}$$

From above equation we have

$$\int_0^{T_1}\Phi(T_1,t)B(t)U(t)dt = \int_0^{T_1}\Phi(T_1,t)B(t)(\frac{1}{T_1}B^{-1}(t)\Phi^{-1}(T_1,t)X_1 - B^T(t)\Phi^T(T_1,t)W^{-1}(0,T_1)\Phi(T_1,0)x_0)dt. \tag{4.37}$$

From the last equation it follows that $U(t)$ given by

$$U(t) = \frac{1}{T_1}B^{-1}(t)\Phi^{-1}(T_1,t)X_1 - B^T(t)\Phi^T(T_1,t)W^{-1}(0,T_1)\Phi(T_1,0)x_0),$$

steers x_0 to X_1 during time-interval $[0,T_1]$. This completes the proof. □

The results of the Theorem 4.3.2 are interesting. However, it has a drawback, that is, the results are applicable only to the class of control systems that has matrix $B(t)$ invertible for all t. It is worthwhile to note for the crisp linear system if the matrix $B(t)$ is invertible, then the system is always controllable and infinitely many controls are possible, for example $u(t) = B(t)^{-1}[\zeta'(t) - A(t)\zeta(t)]$, where $\zeta(t)$ is any differential trajectory satisfying $\zeta(0) = x_0$ and $\zeta(T_1) = x_1$. To overcome this drawback, we obtain some sufficient conditions for the controllability of the system (4.25) for a general $B(t)$ of size $n \times m$. However, it is observed in our analysis that the we do not have complete controllability in this case. The following remark illustrate this fact.

Remark 4.3.3. *It is well known from the crisp control theory that if $U(t) \in \mathbb{R}^m$ then the control $u(\cdot)$ given by*

$$U(t) = B^T(t)\Phi^T(T_1,t)W^{-1}(0,T_1)(x_1 - \Phi(T_1,0)x_0) \tag{4.38}$$

steers the system (4.25) *to a desired state $x_1 \in \mathbb{R}^n$ during time-interval $[0,T_1]$. Clearly* *the control $U(t)$ belongs to $(\mathbb{E}^1)^m$ if the desired state $x_1 \in (\mathbb{E}^1)^n$. However in this case control $U(\cdot)$ may not always steer the system state to the desired state $x_1 \in (\mathbb{E}^1)^n$. Indeed we will show in the following results that only the target states belonging to a subset of $(\mathbb{E}^1)^n$ are controllable.*

Now we introduce the following definition.

Definition 4.3.4. *The fuzzy dynamic system* (4.25) *is said to be quasi-controllable if starting from any initial state x_0 in $\mathbb{R}^n$, there exist a finite time T_1 and admissible controllable subset $\mathbb{E}_0^n$ of $(\mathbb{E}^1)^n$ such that the system state can be brought to any arbitrary state $X_1 \in \mathbb{E}_0^n$ by the fuzzy control inputs $U(t) = B^T(t)\Phi^T(T_1,t)W^{-1}(0,T_1)(X_1 - \Phi(T_1,0)x_0) \in (\mathbb{E}^1)^m$.*

The following theorem describes the quasi-controllable fuzzy states $\mathbb{E}_0^n$ for the system (4.25). Thus, we have the following theorem.

Theorem 4.3.5. *The fuzzy control system* (4.25) *is quasi-controllable if and only if there exists a finite time T_1 such that the controllability Grammian $W(0,T_1)$ is non-singular. And, the admissible controllable state subset $\mathbb{E}_0^n$ is*

$$\mathbb{E}_0^n = \{X_1 \in (\mathbb{E}^1)^n | X_1 = \int_0^{T_1} \Phi(T_1,t)B(t)B^T(t)\Phi^T(T_1,t)W^{-1}(0,T_1)X_1\}.$$

Proof. Let $X_1 \in \mathbb{E}_0^n$ be the desired fuzzy state. The solution of the system (4.25) at time T_1 satisfies

$$x(T_1) = \Phi(T_1,0)x_0 + \int_0^{T_1} \Phi(T_1,t)B(t)U(t)dt. \tag{4.39}$$

By taking $U(t) = B^T(t)\Phi^T(T_1,t)(t)W^{-1}(0,T_1)(X_1 - \Phi(T_1,0)x_0)$, we have

$$\begin{aligned} x(T_1) &= \Phi(T_1,0)x_0 + \int_0^{T_1} \Phi(T_1,t)B(t)B^T(t)\Phi^T(T_1,t)W^{-1}(0,T_1)(X_1 - \Phi(T_1,0)x_0)dt. \\ &= \Phi(T_1,0)x_0 + \int_0^{T_1} \Phi(T_1,t)B(t)B^T(t)\Phi^T(T_1,t)W^{-1}(0,T_1)X_1 dt - \Phi(T_1,0)x_0. \\ &= \int_0^{T_1} \Phi(T_1,t)B(t)B^T(t)\Phi^T(T_1,t)W^{-1}(0,T_1)X_1 dt. \\ &= X_1. \end{aligned} \tag{4.40}$$

Hence the initial state x_0 can be brought to X_1 by the control $U(\cdot)$. $\square$

Remark 4.3.6. *The admissible controllable subset* $\mathbb{E}_0^n$ *can also be expressed in terms of the fixed points of the operator* $\mathcal{T} : (\mathbb{E}^1)^n \to (\mathbb{E}^1)^n$ *defined by*

$$\mathcal{T}x = \int_0^{T_1} \Phi(T_1, t)B(t)B^T(t)\Phi^T(T_1, t)W^{-1}(0, T_1)x.$$

Clearly

$$\mathbb{E}_0^n = \{x \in (\mathbb{E}^1)^n | \mathcal{T}x = x\},$$

that is, $\mathbb{E}_0^n$ *is the set of all fixed points of the operator* $\mathcal{T}$.

The following lemma will be useful in establishing the fixed points of the operator $\mathcal{T}$.

Lemma 4.3.7. *Let* $X_1 \in (\mathbb{E}^1)^n$ *and* $f(t) \in \mathbb{R}^{n\times n}$ *be a matrix valued continuous function. Let*

$$F = \int_0^{T_1} f(t)dt.$$

Assume that F *is invertible and the matrix* $f(t)F^{-1}$ *has non-negative entries for* $t \in [0, T_1]$ *then the following relation holds*:

$$X_1 = \int_0^{T_1} f(t)F^{-1}X_1 dt.$$

Proof. Let $X_1 = (X_{11}, X_{12}, \ldots, X_{1n})$ and $[f(t)F^{-1}]_{ij}$ denotes the entry in i^{th} row and j^{th} column of the matrix $[f(t)F^{-1}]$. Let $[\underline{X_{1j}^\alpha}, \overline{X_{1j}^\alpha}]$ denotes the α-cut of X_{1j} for $j = 1, 2, \ldots, n$. Then

$$\left[\int_0^{T_1} f(t)F^{-1}X_1 dt\right]_\alpha = \Bigg[\int_0^{T_1} \left[\sum_{j=1}^n [f(t)F^{-1}]_{1j}X_{1j}\right]_\alpha dt, \int_0^{T_1} \left[\sum_{j=1}^n [f(t)F^{-1}]_{2j}X_{1j}\right]_\alpha dt, \ldots, \tag{4.41}$$

$$\int_0^{T_1} \left[\sum_{j=1}^n [f(t)F^{-1}]_{nj}X_{1j}\right]_\alpha dt\Bigg]^T.$$

Since $f(t)F^{-1}$ has non-negative entries, therefore we have for $i = 1, 2, \dots, n$

$$\left[\sum_{j=1}^{n}[f(t)F^{-1}]_{ij}X_{1j}\right]_\alpha = \left[\int_0^{T_1}\sum_{j=1}^{n}[f(t)F^{-1}]_{ij}\underline{X_{1j}^\alpha}dt, \int_0^{T_1}\sum_{j=1}^{n}[f(t)F^{-1}]_{ij}\overline{X_{1j}^\alpha}dt\right]. \tag{4.42}$$

By using Eq. (4.42) in (4.41), we have

$$\begin{aligned}
\left[\int_0^{T_1} f(t)F^{-1}X_1 dt\right]_\alpha &= \Big[\left[\int_0^{T_1}\sum_{j=1}^{n}[f(t)F^{-1}]_{1j}\underline{X_{1j}^\alpha}dt, \int_0^{T_1}\sum_{j=1}^{n}[f(t)F^{-1}]_{1j}\overline{X_{1j}^\alpha}dt\right], \dots, \\
&\qquad \left[\int_0^{T_1}\sum_{j=1}^{n}[f(t)F^{-1}]_{nj}\underline{X_{1j}^\alpha}dt, \int_0^{T_1}\sum_{j=1}^{n}[f(t)F^{-1}]_{nj}\overline{X_{1j}^\alpha}dt\right]\Big]^T \\
&= \Big[\int_0^{T_1} f(t)F^{-1}[\underline{X_{11}^\alpha}, \underline{X_{12}^\alpha}, \dots, \underline{X_{1n}^\alpha}]^T dt, \\
&\qquad \int_0^{T_1} f(t)F^{-1}[\overline{X_{11}^\alpha}, \overline{X_{12}^\alpha}, \dots, \overline{X_{1n}^\alpha}]^T dt\Big] \\
&= \left[[\underline{X_{11}^\alpha}, \overline{X_{11}^\alpha}], [\underline{X_{12}^\alpha}, \overline{X_{12}^\alpha}], \dots, [\underline{X_{1n}^\alpha}, \overline{X_{1n}^\alpha}]\right]^T \\
&= [X_1]_\alpha.
\end{aligned}$$

Hence the lemma is proved. □

Remark 4.3.8. *It should be noted if the state $X_1 \in \mathbb{R}^n$, then without any restriction on the entries of $f(t)F^{-1}$ the relation*

$$X_1 = \int_0^T f(t)F^{-1}X_1 dt.$$

holds.

We shall now give an example to show that if some of the entries of $f(t)F^{-1}$ are negative then the results of the Lemma 4.3.7 need not hold.

Example 4.3.9. *Let* $f(t) = \begin{pmatrix} 2t & 1 \\ 0 & 3t^2 \end{pmatrix}$, $X_1 = (\hat{2}, \hat{3}) \in (\mathbb{E}^1)^2$ *with* $[\hat{2}]_\alpha = [1+\alpha, 3-\alpha]$ *and* $[\hat{3}]_\alpha = [2+\alpha, 4-\alpha]$ *and* $t \in [0, 1]$.

Then, $F = \int_0^1 f(t) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. *The matrix* $f(t)F^{-1}$ *is given by*

$$f(t)F^{-1} = \begin{pmatrix} 2t & 1-2t \\ 0 & 3t^2 \end{pmatrix}.$$

We will now show that $[X_1]_\alpha \neq [\int_0^1 f(t)F^{-1}X_1 dt]_\alpha$. *Clearly*

$$[X_1]_\alpha = \begin{pmatrix} [1+\alpha, 3-\alpha] \\ [2+\alpha, 4-\alpha] \end{pmatrix}. \tag{4.43}$$

Now we will compute $[\int_0^1 f(t)F^{-1}X_1 dt]_\alpha$.

$$\begin{aligned} [\int_0^1 f(t)F^{-1}X_1 dt]_\alpha &= \int_0^1 [f(t)F^{-1}X_1]_\alpha dt \\ &= \begin{pmatrix} \int_0^1 [(2t)\hat{2} + (1-2t)\hat{3}]_\alpha dt \\ \int_0^1 [(3t^2)\hat{3}]_\alpha dt \end{pmatrix}. \end{aligned} \tag{4.44}$$

After some more simplification of the terms on $R.H.S.$ *of* (4.44)*, we have*

$$\int_0^1 [f(t)F^{-1}X_1 dt]_\alpha = \begin{pmatrix} \left[-\frac{1}{2} + \frac{3\alpha}{2}, \frac{7}{2} - \frac{3\alpha}{2}\right] \\ [2+\alpha, 4-\alpha] \end{pmatrix}. \tag{4.45}$$

It is clear from Eq. (4.43) *and Eq.* (4.45) *that*

$$[X_1]_\alpha \neq [\int_0^1 f(t)F^{-1}X_1 dt]_\alpha.$$

Hence

$$[X_1] \neq \int_0^1 f(t)F^{-1}X_1 dt.$$

In the following theorem we will give a sufficient condition for the admissible controllable subset $\mathbb{E}_0^n$ to be $(\mathbb{E}^1)^n$.

Theorem 4.3.10. *If there exists a finite time* T_1 *such that the controllability grammian* $W(0, T_1)$ *is non-singular and the matrix* $\Phi(T_1, t)B(t)B^T(t)\Phi^T(T_1, t)W^{-1}(0, T_1)$ *has non negative entries for all* $t \in [0, T_1]$*, then* $x = \mathcal{T}x$ *holds for all* $x \in (\mathbb{E}^1)^n$ *or* *equivalently* $\mathbb{E}_0^n = (\mathbb{E}^1)^n$.

Proof. Let $f(t) = \Phi(T_1, t)B(t)B^T(t)\Phi^T(T_1, t)$ and

$$F = W(0, T_1) = \int_0^{T_1} \Phi(T_1, t)B(t)B^T(t)\Phi^T(T_1, t)dt.$$

By the assumption of the theorem it follows that $f(t)F^{-1}$ have nonnegative entries for all $t \in [0, T_1]$. Now by invoking Lemma 4.3.7, it follows that for every $x \in (\mathbb{E}^1)^n$

$$\begin{aligned} x &= \int_0^{T_1} \Phi(T_1, t)B(t)B^T(t)\Phi^T(T_1, t)W^{-1}(0, T_1)x \qquad (4.46) \\ &= \mathcal{T}x. \end{aligned}$$

That is $\mathbb{E}_0^n = (\mathbb{E}^1)^n$. Hence the theorem. □

Remark 4.3.11. *The conditions of the Theorem 4.3.10 can be easily satisfied for the systems of the form*

$$\dot{x}(t) = a(t)x(t) + b(t)u(t), x(0) = x_0 \in \mathbb{R},$$

where $a(t)$*,* $b(t)$ *are continuous non-negative function on some time interval* $[0, T_1]$ *and* $u(t) \in \mathbb{E}$*. Hence for such systems* $\mathbb{E}_0^1 = (\mathbb{E}^1)$*, that is, the system is completely controllable.*

4.4 Conclusion

In the chapter, we have investigated the controllability property for the linear systems with fuzzy initial conditions and fuzzy inputs. We have established the controllability results by using the levelwise approach and the differential inclusion approach. We also have introduced the concept of 'fuzzy-controllability', a concept weaker than controllability and a procedure to compute the fuzzy-controllable initial states is provided. Our results extend some of the results due to Feng and Hu (2006), Kwun et al. (2008) and Ding and Kandel (2000a). Various illustrative examples are provided to substantiate the concepts and the results obtained.

CHAPTER 5

Controllability of Nonlinear Matrix Lyapunov Systems

5.1 Introduction

The tools of applied mathematics have been explored extensively for tackling control problems in the literature. Many of the real world problems arising in mechanics, biological systems, finance industry and in space applications are control theoretic in nature. Vast literature is available on the controllability of linear and nonlinear systems, for example, George (1995), Joshi and George (1989), Sontag (1998), Zabczyk (2008).

Matrix Lyapunov systems are very important systems and find applications in various engineering applications. These systems are usually regarded as the generalization to usual dynamical systems or control systems. Recently Murty et al. (2006) studied the controllability of the matrix Lyapunov systems

$$\dot{X}(t) = A(t)X(t) + X(t)B(t) + F(t)U(t). \tag{5.1}$$

Furthermore, in Murty and Kumar (2008b) the stability of matrix Lyapunov systems of type (5.1) is investigated. Often the actual system can not be modelled by the linear system of the form (5.1) due to the presence of inherent nonlinearities in the system. Therefore, our aim is to investigate the controllability of nonlinear matrix Lyapunov systems represented by:

$$\dot{X}(t) = A(t)X(t) + X(t)B(t) + F(t)U(t) + G(t, X(t)), \tag{5.2}$$

where $X(t)$ is an $n \times n$ real matrix called state matrix, $U(t)$ is an $m \times n$ real matrix called control matrix and $G(\cdot, \cdot) : \mathbb{R}^+ \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is a nonlinear function. $A(t)$, $B(t)$, $F(t)$ are $n \times n$, $n \times n$ and $n \times m$ real matrices respectively. All of them are assumed to be piecewise continuous functions of t on $[t_0, t_1](0 \leq t_0 < t_1 < \infty)$. Furthermore, the entries in the state matrix $X(t)$ and the control matrix $U(t)$ belong to $L^2([t_0, t_1], \mathbb{R})$. The function G satisfies the 'Caratheodory conditions'; that is, $G(\cdot, x)$ is measurable with respect to t for all $x \in \mathbb{R}^{n\times n}$ and $G(t, \cdot)$ is continuous with respect to x for almost all $t \in [t_0, t_1]$.

Note that under the assumptions $G(t, x) \equiv 0$ the system (5.2) reduces to system (5.1) whose controllability is investigated in Murty et al. (2006). Moreover, if $G(t, x) \equiv 0$ and $B(t) \equiv 0$ then the system (5.2) reduces to linear time-varying control system whose controllability is well established in the literature, for example, Barnett and Cameron (1975), Sontag (1998).

In our work, we investigate complete controllability for nonlinear matrix Lyapunov systems (5.2) using the tools of functional analysis and operator theory. We establish some sufficient conditions for the complete controllability of nonliner matrix Lyapunov systems of type (5.2) involving Lipschitzian and non-Lipschitzian nonlinearities. In case of non-Lipschitzian nonlinearities, we assume that nonlinearities are of monotone type. The organization of the chapter is as follows: In Section 5.2, we state some basic results used in the chapter related to Kronecker products and nonlinear functional analysis. In Section 5.3, nonlinear MLS (5.2) is reduced to a semilinear control systems. In Section 5.4, we reduce the controllability problem in to a solvability problem solution to which in turn leads to sufficient conditions for the controllability of nonlinear MLS. Finally, we conclude the results in Section 5.5 along with few implications for the future research.

5.2 Preliminaries

We introduce some notations that will be used in this chapter. Given any matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $\|A\|_F$ denotes its Frobenius norm and is defined as

$$\|A\|_F := \Big[\sum_{i,j=1}^{n} |a_{ij}|^2\Big]^{\frac{1}{2}};$$

$\|A\|$ denotes the $2-$norm (spectral norm) of A. Given any vector $x \in \mathbb{C}^n$, $\|x\|$ denotes the $2-$norm (Euclidean norm) of x. I_n denotes the $n \times n$ identity matrix. Given any matrix A, $\sum A$ denotes the sum of the absolute values of entries of A.

We start with some basic definitions related to Kronecker products which we shall use in this chapter.

Definition 5.2.1. *(Graham (1981))* Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$ then the Kronecker product of A and B is written as $A \otimes B$ and is defined to be the partitioned matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

which is an $mp \times nq$ matrix and in $\mathbb{C}^{mp\times nq}$.

Definition 5.2.2. Let $A = [a_{ij}] \in \mathbb{C}^{m\times n}$. We denote

$$\hat{A} = \mathrm{Vec}\, A = \begin{bmatrix} A_{.1} \\ A_{.2} \\ \vdots \\ A_{.n} \end{bmatrix}_{mn\times 1}, \quad \text{where} A_{.j} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}, \quad (1 \leq j \leq n).$$

The Kronecker product satisfies the following properties (cf. Graham (1981)):

1. $(A \otimes B)^T = (A^T \otimes B^T)$.
2. $(A \otimes B)^{-1} = (A^{-1} \otimes B^{-1})$.
3. $(A \otimes B)(C \otimes D) = (AC \otimes BD)$, provided the dimensions of various matrices are compatible with matrix product.
4. If $A(t)$ and $B(t)$ are matrices, then

$$\frac{d}{dt}(A(t) \otimes B(t)) = \frac{d}{dt}(A(t)) \otimes B(t) + A(t) \otimes \frac{d}{dt}(B(t)).$$

5. $\mathrm{Vec}(AYB) = (B^T \otimes A)\,\mathrm{Vec}(Y)$.
6. If A and X are matrices of order $n \times n$, then

(i) $\mathrm{Vec}(AX) = (I_n \otimes A)\,\mathrm{Vec}(X)$.

(ii) $\mathrm{Vec}(XA) = (A^T \otimes I_n)\,\mathrm{Vec}(X)$.

Proposition 5.2.3. *The operator $Vec : \mathbb{C}^{m\times n} \to \mathbb{C}^{mn}$ satisfies the following properties:*

1. *$Vec : (\mathbb{C}^{m\times n}, \|\cdot\|) \to (\mathbb{C}^{mn}, \|\cdot\|)$ is a bounded linear operator. Here $\|\cdot\|$ denotes the standard 2-norm. Further, the correspondence $X \to Vec(X)$ is surjective and one-to-one.*

2. *$\|Vec\| \leq \sqrt{n}$.*

3. *$\|Vec^{-1}\| \leq 1$.*

Proof. Clearly, by definition the operator Vec is linear. Since $\|X\|_F \leq \sqrt{n}\|X\|$, for all $X \in \mathbb{C}^{m\times n}$ and $\|X\|_F = \|Vec(X)\|$, therefore we have that

$$\|Vec(X)\| \leq \sqrt{n}\|X\|.$$

This shows that Vec is bounded linear operator with $\|Vec\| \leq \sqrt{n}$. Moreover, given any $x = (x_1, x_2, \ldots, x_{nm}) \in \mathbb{C}^{mn}$, there exists unique matrix $X \in \mathbb{C}^{m\times n}$ defined by

$$X = \begin{bmatrix} x_1 & x_{m+1} & \cdots & x_{(n-1)m+1} \\ x_2 & x_{m+2} & \cdots & x_{(n-1)m+2} \\ \vdots & \vdots & \ddots & \vdots \\ x_m & x_{2m} & \cdots & x_{nm} \end{bmatrix}$$

such that $Vec(X) = x$, this shows that Vec is surjective and one-to-one. Furthermore,

$$\|Vec^{-1}x\| = \|X\| \leq \|x\|.$$

The last equation shows that $\|Vec^{-1}\| \leq 1$. Thus the proof of the proposition follows. □

We will now state some essential definitions and results from nonlinear functional analysis. The following contraction principle will be used in this chapter.

Theorem 5.2.4. *(Joshi (1983)) Let T be a continuous mapping of a Banach space X into itself such that there exists a positive integer $n \geq 1$ such that $\|T^n x - T^n y\| \leq k\|x - y\|$ for all $x, y \in X$ for some positive constant $k < 1$. Then T has a unique fixed point.*

Remark 5.2.5. *When $n = 1$, the above theorem is known as Banach contraction principle. Furthermore, in this case for any arbitrary $y \in X$ the sequence defined by*

$$x_{n+1} = Tx_n + y$$

converges to the unique solution of $x = Tx + y$.

Definition 5.2.6 (Joshi and George (1989))**.** *Let X be a real Banach space. Let "Lip" be the set of all operators $N : X \to X$ which satisfy Lipschitz condition; that is, there exists a constant $\alpha > 0$ such that*

$$\|Nx_1 - Nx_2\| \leq \alpha\|x_1 - x_2\|, \quad \text{for all } x_1, x_2 \in X. \tag{5.3}$$

For $N \in$Lip, we define

$$\|N\|_{\text{Lip}} = \sup_{\substack{x_1, x_2 \in X \\ x_1 \neq x_2}} \frac{\|Nx_1 - Nx_2\|}{\|x_1 - x_2\|}.$$

Definition 5.2.7 (Joshi and George (1989))**.** Let H be a real Hilbert space. Let $\mathcal{M}$ be the set of all operators $N : H \to H$ such that $N \in \mathcal{M}$ if and only if

$$\langle Nx_1 - Nx_2, x_1 - x_2 \rangle \geq \alpha\|x_1 - x_2\|^2,$$

for all $x_1, x_2 \in H$ and α is a constant in $\mathbb{R}$. For $N \in \mathcal{M}$, we define

$$\mu(N) = \inf_{\substack{x_1, x_2 \in H \\ x_1 \neq x_2}} \frac{< Nx_1 - Nx_2, x_1 - x_2 >}{\|x_1 - x_2\|^2}.$$

The operator N is called monotone (strongly monotone) if $\mu(N) \geq 0$ ($\mu(N) > 0$).

Definition 5.2.8. *Let X be a real Banach space and X^* be the dual of X. Let $\mathbb{T} : D(\mathbb{T}) \subset X \to X^*$ be any operator. Then $\mathbb{T}$ is said to be of type (M) if for any se-*

quence $\{x_n\}$ in X converging to x_0 in X with $\{\mathbb{T}x_n\}$ converging weakly to y in X^* and $\limsup_{n\to\infty}(\mathbb{T}x_n)(x_n - x_0) \leq 0$*, we have* $y = \mathbb{T}x_0$.

Definition 5.2.9. *Let $\mathbb{T}$ be the same as in Definition 5.2.8. $\mathbb{T}$ is said to be coercive if*

$$\lim_{\|x\|\to\infty} \frac{(Tx)(x)}{\|x\|} = \infty$$

.

Theorem 5.2.10. *(Joshi and Bose (1985)) Let X be a real Banach space and $T : X \to X^*$ is a mapping of type (M). If T is coercive then the range of T is all of X^*.*

5.3 Reduction of Nonlinear Matrix Lyapunov System in to a Semilinear Control System

By applying Vec operator to equation (5.2), we have the following system:

$$\dot{\psi}(t) = A_1(t)\psi(t) + B_1(t)u(t) + G_1(t, \psi(t)), \tag{5.4}$$

where $\psi(t) = \mathrm{Vec}(X(t))$, $A_1(t) = (B^T \otimes I_n) + (I_n \otimes A)$, $B_1(t) = I_n \otimes F(t)$, $u(t) = \mathrm{Vec}(U(t))$ and $G_1(t, \psi(t)) = \mathrm{Vec}(G(t, X(t)))$.

Definition 5.3.1. The nonlinear matrix Lyapunov system (5.2) is said to be controllable on $[t_0, t_1]$ in the domain of controllability $D \subset \mathbb{R}^{n\times n}$ if for each pair of matrices $X_0, X_1 \in D$, there exists a control $U \in L^2([t_0, t_1]; \mathbb{R}^{m\times n})$ such that the solution of (5.2) together with $X(t_0) = X_0$ also satisfies $X(t_1) = X_1$.

Remark 5.3.2. *If $D = \mathbb{R}^n$ in the above definition, then the system* (5.2) *is called completely or globally controllable. In this chapter, by controllability we always mean the complete controllability.*

Proposition 5.3.3. *The matrix Lyapunov system* (5.2) *is completely controllable if and only if the semilinear system* (5.4) *is completely controllable.*

The proof of the above proposition is trivial as system (5.2) and system (5.4) are equivalent.

Let us consider the corresponding linear system of (5.4), which is given by

$$\dot{\psi}(t) = A_1(t)\psi(t) + B_1(t)u(t) \tag{5.5}$$

Murty et al. (2006) established the necessary and sufficient conditions for the complete controllability of the linear system (5.5).

Theorem 5.3.4. *(Murty et al. (2006)) The system* (5.5) *is completely controllable if and only if the $n^2 \times n^2$ symmetric controllability matrix*

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, s)(I_n \otimes F(s))(I_n \otimes F^T(s))\Phi^T(t_0, s)ds, \tag{5.6}$$

is nonsingular, where $\Phi(t,s) = \Phi_2(t,s) \otimes \Phi_1(t,s)$ is the transition matrix generated by $A_1(t)$ in which Φ_1 and Φ_2 are the transition matrices for systems $\dot{X}(t) = A(t)X(t)$ and $\dot{X}(t) = B^T(t)X(t)$, respectively. In this case the control

$$u(t) = -(I_n \otimes F^T(t))\Phi^T(t_0, t)W^{-1}(t_0, t_1)[\psi_0 - \Phi(t_0, t_1)\psi_1], \tag{5.7}$$

transfers $\psi(t_0) = \psi_0$ to $\psi(t_1) = \psi_1$.

Remark 5.3.5. In the above theorem $W(t_0, t_1)$ can also be defined as follows:

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, s)(I_n \otimes F(s))(I_n \otimes F^T(s))\Phi^T(t_1, s)ds, \tag{5.8}$$

and in this case the control u will be given by

$$u(t) = (I_n \otimes F^T(t))\Phi^T(t_1, t)W^{-1}(t_0, t_1)[\psi_1 - \Phi(t_1, t_0)\psi_0]. \tag{5.9}$$

Remark 5.3.6. $W(t_0, t_1)$, as defined in (5.8), can also be written as CC^*, where $\mathrm{C} : L^2([t_0, t_1]; \mathbb{R}^{mn}) \to \mathbb{R}^{n^2}$ is defined as

$$\mathrm{C}u = \int_{t_0}^{t_1} \Phi(t_1, s)(I_n \otimes F(s))u(s)ds,$$

and $\mathrm{C}^* : \mathbb{R}^{n^2} \to L^2([t_0, t_1]; \mathbb{R}^{mn})$ is the adjoint of C and is defined as follows:

$$(\mathrm{C}^*\psi)(t) = (I_n \otimes F^T(t))\Phi^T(t_1, t)\psi.$$

Proposition 5.3.7. *Let* $\Phi(t,s)$ *be the same as in Theorem 5.3.4. Then the solution of* (5.4) *with initial condition* $\psi(t_0)=\psi_0$ *is given by the following Volterra-type integral equation*:

$$\psi(t)=\Phi(t,t_0)\psi_0+\int_{t_0}^{t}\Phi(t,s)((I_n\otimes F(s))u(s)+G_1(s,\psi(s)))ds. \tag{5.10}$$

The proof of the above proposition can be obtained by using the standard technique of the variation of parameter.

Note that we are interested in the complete controllability of (5.2) which, by Proposition 5.3.3, is equivalent to the complete controllability of (5.4); that is, the domain of controllability is $\mathbb{R}^{n^2}$. Furthermore, the controllability results for nonlinear system (5.4) will mainly depend on the controllability results of corresponding linear system (5.5). Therefore, we assume that the linear system (5.5) is completely controllable.

5.4 Controllability Results

In this Section, we discuss the complete controllability results for the nonlinear matrix Lyapunov system (5.2). First we convert the controllability problem in to a solvability problem which is solved by using results from operator theory and nonlinear functional analysis, that in turn provide solution for the original controllability problem.

5.4.1 Reduction of Controllability Problem to a Solvability Problem

Now we shall discuss the controllability of system (5.4) in terms of the solvability of an equivalent feedback system of the form

$$\begin{aligned} e_1 &= u_1 - S_2e_2 \\ e_2 &= u_2 + S_1e_1, \end{aligned} \tag{5.11}$$

for some appropriate operator $S_1 : X_1 \to X_2$ and $S_2 : X_2 \to X_1$, where X_1 and X_2 are some suitable Banach spaces. Now we will define the solvability of the system (5.11).

Definition 5.4.1. Let X_1 and X_2 be real Banach spaces. The feedback system (5.11) is said to be globally solvable if for every $(u_1, u_2) \in X_1 \times X_2$, there exists a solution $(e_1, e_2) \in (X_1 \times X_2)$ of (5.11). If this solution is unique then it is said to be uniquely globally solvable.

The following lemma describes the conditions when the system (5.11) is uniquely globally solvable. Furthermore, it provides an iterative scheme that converge to the unique solution (e_1, e_2) of the system starting from any arbitrary initial condition, say, $e_1^0 \in X_1$.

Lemma 5.4.2. *Let $S_1 : X_1 \to X_2$ and $S_2 : X_2 \to X_1$ belong to the class* Lip. *If $\|S_1\|_{\text{Lip}}\|S_2\|_{\text{Lip}} < 1$, then the system* (5.11) *is uniquely globally solvable. Moreover the iterates $[e_1^{(n)}]$, $[e_2^{(n)}]$ defined by*

$$\begin{aligned} e_1^{(n+1)} &= u_1 - S_2 e_2^{(n)} \\ e_2^{(n)} &= u_2 + S_1 e_1^{(n)} \end{aligned}$$

converge to the unique solution $(e_1, e_2) \in (X_1 \times X_2)$ starting from arbitrary $e_1^{(0)} \in X_1$.

Proof. It is clear that the coupled system (5.11) is equivalent to the following system

$$\begin{aligned} e_1 &= u_1 - S_2(u_2 + S_1 e_1), \\ e_2 &= u_2 + S_1 e_1. \end{aligned} \tag{5.12}$$

Therefore, solvability of the system (5.11) is equivalent to the solvability of the system (5.12). Define $W : X_1 \to X_2$ by $We_1 = u_2 + S_1 e_1$. Then the above pair of system is equivalent to the following system of equations:

$$e_1 = u_1 - S_2 W e_1, \tag{5.13}$$

$$e_2 = W e_1. \tag{5.14}$$

Since given any $e_1^1, e_1^2 \in X_1$, we have

$$\|We_1^1 - We_1^2\| = \|S_1 e_1^1 - S_1 e_1^2\|.$$

From above equation, it follows that $W \in \text{Lip}$ with $\|W\|_{Lip} = \|S_1\|_{Lip}$. Now by the hypothesis of the lemma, we have

$$\|S_2 W\|_{Lip} \leq \|S_2\|_{Lip}\|W\|_{Lip} = \|S_2\|_{Lip}\|S_1\|_{Lip} < 1.$$

Now by a variation of Banach contraction principle (see Remark 5.2.5), it follows that for every $u_1 \in X_1$, there exists a unique solution $e_1 \in X_1$ of (5.13). Now take $e_2 = We_1 = u_2 + S_1 e_1$. Clearly e_2 is also unique. Thus, the feedback system (5.11) is uniquely globally solvable.

Furthermore, the contraction mapping principle implies that the iterates defined by

$$e_1^{(n+1)} = u_1 - S_2(u_2 + S_1 e_1^{(n)})$$

converges to the unique solution $e_1 \in X_1$ of (5.13) starting from any initial condition $e_1^{(0)} \in X_1$. The above equation implies that the iterates

$$\begin{aligned} e_1^{(n+1)} &= u_1 - S_2 e_2^{(n)} \\ e_2^{(n)} &= u_2 + S_1 e_1^{(n)}, \end{aligned}$$

converge to the unique solution $(e_1, e_2) \in X_1 \times X_2$ of (5.11). □

Suppose that the system (5.4) is completely controllable on $[t_0, t_1]$. That is, there exists a control u in $L^2([t_0, t_1]; \mathbb{R}^{mn})$ which steers the initial state $\psi_0 \in \mathbb{R}^{n^2}$ of system (5.4) to the desired final state $\psi_1 \in \mathbb{R}^{n^2}$. Then according to Proposition 5.3.7 we have:

$$\psi_1 = \psi(t_1) = \Phi(t_1, t_0)\psi_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B_1(\tau)u(\tau)d\tau + \int_{t_0}^{t_1} \Phi(t_1, \tau)G_1(\tau, \psi(\tau))d\tau.$$

That is,

$$\psi_1 - \Phi(t_1, t_0)\psi_0 - \int_{t_0}^{t_1} \Phi(t_1, \tau)G_1(\tau, \psi(\tau))d\tau = \int_{t_0}^{t_1} \Phi(t_1, \tau)B_1(\tau)u(\tau)d\tau.$$

Consider now the integral equation

$$\begin{aligned}\psi(t) = \Phi(t, t_0)\psi_0 + \int_{t_0}^{t} \Phi(t, \tau)G_1(\tau, \psi(\tau))d\tau \\ + \int_{t_0}^{t} \Phi(t, \tau)B_1(\tau)\Big(\mathrm{C}^*(\mathrm{CC}^*)^{-1}\big[\psi_1 - \Phi(t_1, t_0)\psi_0 \\ - \int_{t_0}^{t_1} \Phi(t_1, \tau)G_1(\tau, \psi(\tau))d\tau\big]\Big)(\tau)d\tau.\end{aligned} \tag{5.15}$$

Suppose that (5.15) is solvable for some ψ. Then it can be verified that $\psi(t_0) = \psi_0$ and $\psi(t_1) = \psi_1$. This implies the control u which steers the system (5.4) from ψ_0 to ψ_1 is given by

$$u(t) = (\mathrm{C}^*(\mathrm{CC}^*)^{-1}[\psi_1 - \Phi(t_1, t_0)\psi_0 - \int_{t_0}^{t_1} \Phi(t_1, \tau)G_1(\tau, \psi(\tau))d\tau])(t).$$

Hence the controllability of nonlinear system (5.4) is equivalent to the solvability of coupled equations:

$$\begin{aligned}\psi(t) &= \Phi(t, t_0)\psi_0 + \int_{t_0}^{t} \Phi(t, \tau)G_1(\tau, \psi(\tau))d\tau + \int_{t_0}^{t} \Phi(t, \tau)B_1(\tau)u(\tau)d\tau, \\ u(t) &= (\mathrm{C}^*(\mathrm{CC}^*)^{-1}[\psi_1 - \Phi(t_1, t_0)\psi_0 - \int_{t_0}^{t_1} \Phi(t_1, \tau)G_1(\tau, \psi(\tau))d\tau])(t).\end{aligned} \tag{5.16}$$

Let $X_1 = L^2([t_0, t_1]; \mathbb{R}^{mn})$, $X_2 = L^2([t_0, t_1]; \mathbb{R}^{n^2})$. Define operators $K, N : X_2 \to X_2$, $H : X_1 \to X_2$ and $R : X_2 \to X_1$ as follows:

$$(K\psi)(t) = \int_{t_0}^{t} \Phi(t, \tau)\psi(\tau)d\tau, \quad (N\psi)(t) = G_1(t, \psi(t)),$$

$$(Hu)(t) = \int_{t_0}^{t} \Phi(t, \tau)B_1(\tau)u(\tau)d\tau,$$

$$(R\psi)(t) = (\mathrm{C}^*(\mathrm{CC}^*)^{-1} \int_{t_0}^{t_1} \Phi(t_1, \tau)\psi(\tau)d\tau)(t).$$

With this notation, equations (5.16) can be written as a pair of operator equations

$$\begin{aligned} \psi &= u_0 + KN\psi + Hu \\ u &= u_1 - RN\psi, \end{aligned} \tag{5.17}$$

where $u_0(t) = \Phi(t, t_0)\psi_0$ and $u_1(t) = (\mathrm{C}^*(\mathrm{CC}^*)^{-1}[\psi_1 - \Phi(t_1, t_0)\psi_0])(t)$. Without loss of generality ψ_0 can be taken as 0 as indicated in the following theorem.

Theorem 5.4.3. *The system* (5.4) *is globally controllable if and only if for* $x_1 \in \mathbb{R}^{n^2}$ *there is a control* $u \in L^2([t_0, t_1]; \mathbb{R}^{mn})$ *which steers* 0 *to* x_1.

The proof of the above theorem follows by the same argument as in (Russell, 1979, Proposition 2.2). Now using the above theorem the coupled system (5.17) can be written as follows:

$$\begin{aligned} \psi &= KN\psi + Hu, \\ u &= u_1 - RN\psi, \end{aligned} \tag{5.18}$$

where $u_1 = \mathrm{C}^*(\mathrm{CC}^*)^{-1}\psi_1$. Thus, the nonlinear system (5.4) is controllable if and only if the above pair of operator equations (5.18) is solvable. We now introduce operators $\mathcal{M}_1 : X_1 \to X_2$ and $\mathcal{M}_2 : X_2 \to X_1$ as follows:

$$\mathcal{M}_1 = (I - KN)^{-1}H, \quad \mathcal{M}_2 = RN.$$

Now the following lemma is immediate.

Lemma 5.4.4. *If the operator* $(I - KN)$ *is invertible then the controllability of the system* (5.4) *is equivalent to the solvability of the feed-back system*

$$\begin{aligned} \psi &= \mathcal{M}_1 u, \\ u &= u_1 - \mathcal{M}_2\psi. \end{aligned} \tag{5.19}$$

5.4.2 Controllability Results with Lipschitzian Nonlinearities

Now we make the following assumptions:

(A1) Let $b = \sup_{t_0 \le t \le t_1} \|B_1(t)\|$ and the transition matrix $\Phi(t, s)$ is such that $\|\Phi(t, s)\| \le$

$h(t,s)$, where $h(\cdot,\cdot) : [t_0, t_1] \times [t_0, t_1] \to \mathbb{R}^+$ is a function satisfying

$$\Big[\int_{t_0}^{t_1}\int_{t_0}^{t} h^2(t,s)\,ds\,dt\Big]^{\frac{1}{2}} = k < \infty.$$

(A2) The function $G : [t_0, t_1] \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ satisfies the 'Caratheodory conditions'. Further, G satisfies Lipschitz condition with Lipschitz constant α. That is,

$$\|G(t,x) - G(t,y)\| \leq \alpha\|x - y\|.$$

Lemma 5.4.5. *Under Assumptions* (A1)–(A2)*, the bounds for* $\|K\|$, $\|H\|$ *and* $\|R\|$ *are estimated as* $\|K\| \leq k$, $\|H\| \leq bk \triangleq h$ *and* $\|R\| \leq bk_1^2 c \triangleq \gamma$ *where* $c = \|(CC^*)^{-1}\|$ *and* $k_1 = [\int_{t_0}^{t_1} h^2(t_1,s)ds]^{\frac{1}{2}}$.

Proof. We will show that $\|K\| \leq k$.

$$\begin{aligned}
\|Kx\|_{X_2}^2 &= \int_{t_0}^{t_1} \|(Kx)(t)\|^2 dt \\
&= \int_{t_0}^{t_1} \|\int_{t_0}^{t} \Phi(t,\tau)x(\tau)d\tau\|^2 dt \\
&\leq \int_{t_0}^{t_1} \Big(\int_{t_0}^{t} \|\Phi(t,\tau)x(\tau)\|d\tau\Big)^2 dt.
\end{aligned}$$

By using Hölder's inequality on the last expression, we have

$$\begin{aligned}
\|Kx\|_{X_2}^2 &\leq \int_{t_0}^{t_1} \Big(\int_{t_0}^{t} \|\Phi(t,\tau)\|^2 d\tau\Big)\Big(\int_{t_0}^{t} \|x(\tau)\|^2 d\tau\Big) dt. \\
\|Kx\|_{X_2}^2 &\leq \Big(\int_{t_0}^{t_1}\int_{t_0}^{t} h^2(t,\tau)d\tau dt\Big)\|x\|_{X_2}^2.
\end{aligned}$$

Now $\|K\| \leq k$ follows from the last inequality. In order to compute norm of $\|H\|$ consider the following:

$$\begin{aligned}
\|Hu\|_{X_2}^2 &= \int_{t_0}^{t_1} \|(Hu)(t)\|^2 dt \\
&= \int_{t_0}^{t_1} \|\int_{t_0}^{t} \Phi(t,\tau)B_1(\tau)u(\tau)d\tau\|^2 dt \\
&\leq \int_{t_0}^{t_1} \Big(\int_{t_0}^{t} \|\Phi(t,\tau)B_1(\tau)u(\tau)\|d\tau\Big)^2 dt.
\end{aligned}$$

Again by using Hölder's inequality on above term, we have

$$\|Hu\|_{X_2}^2 \leq \int_{t_0}^{t_1} \Big(\int_{t_0}^{t} \|\Phi(t,\tau)\|^2 d\tau \Big) \Big(\int_{t_0}^{t} \|B_1(\tau)u(\tau)\|^2 d\tau \Big) dt.$$
$$\|Hu\|_{X_2}^2 \leq \Big(\int_{t_0}^{t_1} \int_{t_0}^{t} h^2(t,\tau) d\tau dt \Big) b^2 \|u\|_{X_1}^2.$$
$$\|Hu\|_{X_2} \leq bk\|u\|_{X_1}.$$

This shows that $\|H\| \leq bk$. Now we will show that $\|R\| \leq bk_1^2 c$.

$$\begin{aligned}
\|Rx\|_{X_1}^2 &= \int_{t_0}^{t_1} \|(Rx)(t)\|^2 dt \\
&= \int_{t_0}^{t_1} \|(\mathrm{C}^*(\mathrm{CC}^*)^{-1} \int_{t_0}^{t_1} \Phi(t_1,\tau)x(\tau)d\tau)(t)\|^2 dt \\
&= \int_{t_0}^{t_1} \|(B_1^T(t)\Phi^T(t_1,t)(\mathrm{CC}^*)^{-1} \int_{t_0}^{t_1} \Phi(t_1,\tau)x(\tau)d\tau)(t)\|^2 dt. \\
&\leq \int_{t_0}^{t_1} \|(B_1^T(t)\Phi^T(t_1,t)\|^2 dt \|(\mathrm{CC}^*)^{-1}\|^2 \Big(\int_{t_0}^{t_1} h^2(t_1,\tau)d\tau \Big) \|x\|_{X_2}^2 \\
&\leq c^2 b^2 \Big(\int_{t_0}^{t_1} h^2(t_1,\tau)d\tau \Big)^2 \|x\|_{X_2}^2.
\end{aligned}$$

The last inequality implies that

$$\|Rx\|_{X_1} \leq cb \Big(\int_{t_0}^{t_1} h^2(t_1,\tau)d\tau \Big) \|x\|_{X_2}.$$
$$\|Rx\|_{X_1} \leq cbk_1^2 \|x\|_{X_2}.$$

This shows that norm of $\|R\| \leq bk_1^2 c$. Hence the lemma. □

Lemma 5.4.6. *Under Assumption* (A1)–(A2) *the nonlinear operator* N *is Lipschitz continuous and bounded from* X_2 *into itself with Lipschitz constant* $\beta = \sqrt{n}\alpha$.

Proof. We shall prove this lemma by using the following well known matrix norm inequality (see (Hespanha, 2009, p. 64))

$$\|A\| \leq \|A\|_F \leq \sqrt{n}\|A\| \tag{5.20}$$

for any matrix $A \in \mathbb{R}^{n\times n}$. Given any $x_1, x_2 \in \mathbb{R}^{n\times n}$, let $\psi_1 = \mathrm{Vec}(x_1)$, $\psi_2 = \mathrm{Vec}(x_2)$.

By using (5.20) the following relation holds for all $t \in [t_0, t_1]$

$$\|G(t,x_1) - G(t,x_2)\| \leq \|G(t,x_1) - G(t,x_2)\|_F \leq \sqrt{n}\|G(t,x_1) - G(t,x_2)\|. \quad (5.21)$$

Furthermore, we have

$$\|G_1(t,\psi_1) - G_1(t,\psi_2)\| = \|G(t,x_1) - G(t,x_2)\|_F. \quad (5.22)$$

By using (5.22) in (5.21) we have

$$\|G(t,x_1) - G(t,x_2)\| \leq \|G_1(t,\psi_1) - G_1(t,\psi_2)\| \leq \sqrt{n}\|G(t,x_1) - G(t,x_2)\|. \quad (5.23)$$

Let $\psi_1(\cdot)$, $\psi_2(\cdot) \in X_2$ be arbitrary. Using relation (5.23) and Assumption (A2) we can show that

$$\|(N\psi_1)(t) - (N\psi_2)(t)\| \leq \sqrt{n}\alpha\|\psi_1(t) - \psi_2(t)\|. \quad (5.24)$$

The last inequality will in turn implies that N is Lipschitz continuous with Lipschitz constant $\sqrt{n}\alpha$. □

Theorem 5.4.7. *Let β be the Lipschitz constant for nonlinear operator N. Then the operator $I - KN$ is invertible if Assumptions* (A1)–(A2) *hold along with the condition $k\beta < 1$. Furthermore, $(I - KN)^{-1}$ is Lipschitz continuous with Lipschitz constant $\frac{1}{1-k\beta}$.*

Proof. First we show that under the assumptions of the theorem the operator $KN : X_2 \to X_2$ is a contraction. Since,

$$\begin{aligned}\|KN(x_1) - KN(x_2)\|_{X_2} &\leq k\|N(x_1) - N(x_2)\|_{X_2} \\ &\leq k\beta\|x_1 - x_2\|_{X_2} \\ &< \|x_1 - x_2\|_{X_2}.\end{aligned}$$

Hence KN is a contraction. Now by using Banach contraction principle it can be shown that for each fixed $y \in X_2$ the equation $(I - KN)x = y$ has the unique solution; say

x_y. Indeed x_y is the unique limit of the iterates

$$x_{n+1} = KNx_n + y.$$

Now the correspondence $(I - KN)^{-1} : X_2 \to X_2$ given by

$$(I - KN)^{-1}y = x_y \tag{5.25}$$

is well defined. Hence $(I - KN)^{-1}$ is invertible. Furthermore,

$$\begin{aligned}
\|(I - KN)^{-1}(y_1) - (I - KN)^{-1}(y_2)\| &= \|x_{y_1} - x_{y_2}\| \\
&= \|KNx_{y_1} + y_1 - KNx_{y_2} - y_2\| \\
&\leq \|K\|\|Nx_{y_1} - Nx_{y_2}\| + \|y_1 - y_2\| \\
&\leq k\beta\|x_{y_1} - x_{y_1}\| + \|y_1 - y_2\|.
\end{aligned}$$

Hence,

$$(1 - k\beta)\|x_{y_1} - x_{y_2}\| \leq \|y_1 - y_2\|. \tag{5.26}$$

That is,

$$\|(I - KN)^{-1}(y_1) - (I - KN)^{-1}(y_2)\| \leq \frac{1}{(1 - k\beta)}\|(y_1 - y_2)\|. \tag{5.27}$$

Equation (5.27) shows that $(I - KN)^{-1}$ is Lipschitz continuous with Lipschitz constant $\frac{1}{1-k\beta}$. □

Now we have the following theorem which describes the complete controllability of nonlinear matrix Lyapunov system (5.2)

Theorem 5.4.8. *Let β be the Lipschitz constant for the nonlinear operator N. Suppose that the linear system* (5.5) *is controllable and Assumptions* (A1)–(A2) *are satisfied with $k\beta < 1$ and $(\frac{\beta}{1-k\beta})\gamma h < 1$ then*

1. *System* (5.2) *is completely controllable.*

2. *The control matrix $U(t) \in \mathbb{R}^{m\times n}$ steering the zero initial state to desired state $X_1 \in \mathbb{R}^{n\times n}$ during time interval $[t_0, t_1]$ can be approximated by the iterates*

$U^{(n)}(t)$ defined by

$$Vec(U^{(n)})(t) = (\mathrm{C}^*(\mathrm{CC}^*)^{-1}[Vec(X_1) - \int_{t_0}^{t_1} \Phi(t_1, s)G_1(s, Vec(X^n)(s))ds])(t). \tag{5.28}$$

The state matrix approximation $X^{(n+1)}(t)$ at $n+1^{th}$ stage is given by the approximation scheme $X_j^{(n+1)}(t)$

$$Vec(X_{j+1}^{(n+1)})(t) = \int_{t_0}^{t} \Phi(t, s)G_1(s, Vec(X_j^{(n+1)})(s))ds + \int_{t_0}^{t} \Phi(t, s)B_1(s)Vec(U^{(n)})(s)ds. \tag{5.29}$$

Proof. Since $k\beta < 1$, therefore by using Lemma 5.4.7 the operator $I-KN$ is invertible. Also, by Lemma 5.4.4 the controllability of semilinear system (5.4) is equivalent to the solvability of system (5.19). Furthermore,

$$\|\mathcal{M}_1\|_{\mathrm{Lip}} = \|(I - KN)^{-1}H\|_{\mathrm{Lip}} < \frac{h}{1 - k\beta}, \quad \|\mathcal{M}_2\|_{\mathrm{Lip}} = \|RN\|_{\mathrm{Lip}} < \gamma\beta. \tag{5.30}$$

Therefore, by the assumption $(\frac{\beta}{1-k\beta})\gamma h < 1$, we have $\|\mathcal{M}_1\|_{\mathrm{Lip}}\|\mathcal{M}_2\|_{\mathrm{Lip}} < 1$. Since (5.19) is special form of (5.11) and system (5.19) satisfies $\|\mathcal{M}_1\|_{\mathrm{Lip}}\|\mathcal{M}_2\|_{\mathrm{Lip}} < 1$. Hence Lemma 5.4.2 implies that the feedback system (5.19) is uniquely globally solvable, which in turn implies that the nonlinear system (5.4) is completely controllable. Now Proposition 5.3.3 implies that nonlinear matrix Lyapunov system (5.2) is completely controllable.

Furthermore, from Lemma 5.4.2, it follows that starting from any initial state $\psi^0 \in X_1$ the iterates

$$\psi^{(n+1)} = \mathcal{M}_1 u^{(n)}, \tag{5.31}$$

$$u^{(n)} = u_1 - \mathcal{M}_2 \psi^{(n)}, \tag{5.32}$$

where $u_1 = \mathrm{C}^*(\mathrm{CC}^*)^{-1}\psi_1$ and ψ_1 is the desired final state at time t_1, converges to the unique solution of the feedback system (5.19). From (5.32) it follows that

$$
\begin{aligned}
u^{(n)}(t) &= u_1 - RN\psi^{(n)} \\
&= (\mathrm{C}^*(\mathrm{CC}^*)^{-1}[\psi_1 - \int_{t_0}^{t_1} \Phi(t_1, s)G_1(s, \psi^{(n)})(s)ds])(t).
\end{aligned}
$$

Substituting $u(t) = Vec(U(t))$ and $\psi(t) = Vec(X(t))$ in the above relation, the iterates for the control $U(t) \in \mathbb{R}^{m\times n}$ that steers the initial state $X_0 = 0 \in \mathbb{R}^{n\times n}$ at t_0 to a desired state $X_1 = Vec^{-1}(\psi_1)$ are given by

$$
Vec(U^{(n)})(t) = (\mathrm{C}^*(\mathrm{CC}^*)^{-1}[Vex(X_1) - \int_{t_0}^{t_1} \Phi(t_1, s)G_1(s, Vec(X^{(n)})(s))ds])(t). \tag{5.33}
$$

This establishes (5.28).

The state matrix approximation at $(n+1)^{th}$ stage is given by (5.31).

$$
\begin{aligned}
\psi^{(n+1)} &= \mathcal{M}_1 u^{(n)} \\
&= (I - KN)^{-1} H u^{(n)}.
\end{aligned}
$$

From the above equation, we have

$$
\psi^{(n+1)} = KN\psi^{(n+1)} + Hu^{(n)}.
$$

Since by hypothesis of the theorem, KN is a contraction. Therefore, $\psi^{(n+1)}$ is the unique limit of the iterates

$$
\psi_{j+1}^{(n+1)} = KN\psi_j^{(n+1)} + Hu^{(n)}.
$$

From the above equation, we have

$$
\psi_{j+1}^{(n+1)}(t) = \int_{t_0}^{t} \Phi(t, s)G_1(s, \psi_j^{(n+1)}(s))ds + \int_{t_0}^{t} \Phi(t, s)B_1(s)u^{(n)}(s)ds \tag{5.34}
$$

Substituting $u(t) = Vec(U(t))$ and $\psi(t) = Vec(X(t))$ in equation (5.35), we have

$$Vec(X_{j+1}^{(n+1)})(t) = \int_{t_0}^{t} \Phi(t,s)G_1(s, Vec(X_j^{(n+1)})(s))ds + \int_{t_0}^{t} \Phi(t,s)B_1(s)Vec(U^{(n)})(s)ds \tag{5.35}$$

This establishes (5.29). Thus the proof of theorem follows. □

Theorem 5.4.9. *Suppose that the linear system* (5.5) *is controllable and Assumptions* (A1)–(A2) *hold along with $h(s,t) = M$ (where M being a positive constant). Furthermore, $e^{M\beta(t_1-t_0)}\gamma h\beta < 1$, where β is the Lipschitz constant for G_1. Then the conclusions of Theorem 5.4.8 hold.*

Proof. We will first show that the operator $(I - KN)^{-1}$ is Lipschitz continuous with $\|(I - KN)^{-1}\|_{\text{Lip}} \leq e^{M\beta(t_1-t_0)}$. Let $y \in X_2$ be arbitrary. We will start by showing that $(I - KN)^{-1}(y)$ is well defined. Consider the Volterra type integral equation

$$x(t) = \int_{t_0}^{t} \Phi(t,\tau)G_1(\tau, x(\tau))d\tau + y(t). \tag{5.36}$$

Define the following iterates

$$x_0(t) = y(t) \quad \forall t \in [t_0, t_1]. \tag{5.37}$$

$$x_{n+1}(t) = y(t) + \int_{t_0}^{t} \Phi(t,\tau)G_1(\tau, x_n(\tau))d\tau, \quad n = 0, 1, 2 \ldots \tag{5.38}$$

By using Lipschitz continuity of $G_1(t,x)$ and the boundedness of $\Phi(t,\tau)$ in $[t_0, t_1]$, it can shown that the iterates $\{x_n\}$ converges to the solution of (5.36). Furthermore, by applying Gronwall's inequality (Zabczyk, 2008, p.92), uniqueness of the solution of integral equation (5.36) can be easily proved. This in turn shows that the operator $(I - KN)^{-1}$ is well defined. Furthermore, given any $y_1, y_2 \in X_2$, let $(I - KN)^{-1}(y_1) =$

x_1 and $(I - KN)^{-1}(y_2) = x_2$. Then, we have

$$\begin{aligned}
&\|(I - KN)^{-1}(y_1)(t) - (I - KN)^{-1}(y_2)(t)\| \\
&= \|x_1(t) - x_2(t)\| \\
&= \|y_1(t) - y_2(t)\| + \Big\| \int_{t_0}^{t} \Phi(t,\tau)[G_1(\tau, x_1(\tau)) - G_1(\tau, x_2(\tau))]d\tau \Big\| \\
&\leq \|y_1(t) - y_2(t)\| + \int_{t_0}^{t} M\beta \|x_1(\tau) - x_2(\tau)\| d\tau.
\end{aligned}$$

Now again by applying Gronwall's inequality, we have

$$\|x_1(t) - x_2(t)\| \leq e^{M\beta(t_1 - t_0)} \|y_1(t) - y_2(t)\|.$$

Hence we have,

$$\|(I - KN)^{-1}(y_1) - (I - KN)^{-1}(y_2)\|_{X_2} \leq e^{M\beta(t_1 - t_0)} \|y_1 - y_2\|_{X_2}.$$

Thus, we have shown that $\|(I - KN)^{-1}\|_{\text{Lip}} \leq e^{M\beta(t_1 - t_0)}$. Now it follows that

$$\|\mathcal{M}_1\|_{\text{Lip}} = \|(I - KN)^{-1}H\|_{\text{Lip}} < e^{M\beta(t_1 - t_0)}h, \quad \|\mathcal{M}_2\|_{\text{Lip}} = \|RN\|_{\text{Lip}} < \gamma\beta.$$

By the given condition $e^{M\beta(t_1 - t_0)}\gamma h\beta < 1$, it follows that $\|\mathcal{M}_1\|_{\text{Lip}}\|\mathcal{M}_2\|_{\text{Lip}} < 1$. Now the remaining part of the proof is obvious and is same as in Theorem 5.4.8. □

5.4.3 Controllability Results with non-Lipschitzian Nonlinearities

In this section we establish the controllability results for the nonlinear matrix Lyapunov system (5.2) with non-Lipschitzian nonlinearity. In particular we require monotonicity type of condition on the nonlinear term G. Such assumptions are quite reasonable because practically we have situations where the derivatives of the nonlinearities are bounded below by a constant.

We will use the following lemma which guarantees the solvability of the feedback system (5.11).

Lemma 5.4.10. *Let X_1 and X_2 be Hilbert spaces. Let $S_1 : X_1 \to X_2$, $S_2 : X_2 \to X_1$*

be the operators satisfying the following conditions:

(i) S_1 *is compact, continuous and satisfy the growth condition of the type*

$$\|S_1 e_1\| \leq \overline{s_1} + s_1\|e_1\|, \quad \forall e_1 \in X_1,\ and s_1, \overline{s_1} > 0.$$

(ii) S_2 *is continuous and satisfy the growth condition of the type*

$$\|S_2 e_2\| \leq \overline{s_2} + s_2\|e_2\|, \quad \forall e_2 \in X_2,\ s_2, \overline{s_2} > 0.$$

If $(1 - s_1 s_2) > 0$ *then the feedback system* (5.11) *is solvable.*

Proof. Define the operator $T : X_1 \to X_1$ by

$$Te_1 = S_2(u_2 + S_1 e_1).$$

Then the feedback system (5.11) is solvable if and only if

$$(I + T)e_1 = u_1 \tag{5.39}$$

is solvable in X_1. It can be easily shown that T is compact and satisfies the growth condition

$$\|Te_1\| \leq (\overline{s_2} + s_2\|u_2\| + s_2\overline{s_1}) + s_2 s_1\|e_1\|.$$

Since $[I + T]$ is compact and continuous perturbation of the identity map, it is of type (M) (refer to Theorem 3.6.7 Joshi and Bose (1985)). Further,

$$\begin{aligned} < (I + T)e_1, e_1 > &= < e_1, e_1 > + < Te_1, e_1 > \\ &\geq \|e_1\|^2 - \|Te_1\|\|e_1\| \\ &\geq [(1 - s_1 s_2)\|e_1\| - (\overline{s_2} + s_2\|u_2\| + s_2\overline{s_1})]\|e_1\|. \end{aligned}$$

Since $(1 - s_1 s_2) \geq 0$, it follows that

$$\lim_{\|e_1\| \to \infty} \frac{< (I + T)e_1, e_1 >}{\|e_1\|} = \infty.$$

The above equation shows that $(I + T)$ is coercive and hence by Theorem 5.2.10, it follows that range of $(I + T)$ is whole of X_1. This implies that equation (5.39) is solvable which in turn implies that feedback system (5.11) is solvable. $\square$

Let us now assume that the system (5.2) satisfies the following assumptions:

(B1) There exists a positive constant μ such that the matrix $A_1(t)$ satisfies

$$< -A_1(t)\psi, \psi >\geq \mu\|\psi\|^2.$$

(B2) The nonlinear function $-G$ is monotone. In fact $-G$ should satisfy a weaker condition than monotonicity as given below. Given any $x_1, x_2 \in \mathbb{R}^{n\times n}$

$$< (G(t,x_1) - G(t,x_2))e_j, (x_1 - x_2)e_j >\leq 0, 1 \leq j \leq n,$$

where $[e_j]$ denotes the canonical basis in $\mathbb{R}^n$.

(B3) G also satisfies a growth condition of the form

$$\|G(t,x)\| \leq d(t) + w\|x\|,$$

for all $(t,x) \in [t_0,t_1] \times \mathbb{R}^{n\times n}$, $d(\cdot) \in L^2([t_0,t_1];\mathbb{R})$ and $w > 0$.

Theorem 5.4.11. *Under the assumptions* (B1)–(B3)*, the operator* $(I - KN)^{-1}$ *exists and continuous. Furthermore, it satisfies a growth condition of the type*

$$\|(I - KN)^{-1}y\| \leq \frac{d\sqrt{n}}{\mu} + \big(\frac{w\sqrt{n}}{\mu} + 1\big)\|y\|, \tag{5.40}$$

where $d = \|d(\cdot)\|_{L^2[t_0,t_1]}$.

Proof. Assumption (B2) implies that $< (G_1(t,\psi_1) - G_1(t,\psi_2)), (\psi_1 - \psi_2) >\leq 0$ for every ψ_1 and $\psi_2 \in \mathbb{R}^{n^2}$. Furthermore, inequality (5.23) together with assumption $[B3]$ implies that

$$\|G_1(t,\psi)\| \leq \sqrt{n}(d(t) + w\|\psi\|),$$

for all $(t,\psi) \in [t_0,t_1] \times \mathbb{R}^{n^2}$. Now all the requirements of Theorem 5.1 of Joshi and George (1989) are satisfied. A careful trace of the Theorem 5.1 of Joshi and George (1989) will prove the theorem. □

Theorem 5.4.12. *Suppose that the linear system* (5.5) *is controllable and the assumptions* (A1) *and* (B1)–(B3) *are satisfied. If* $[1-(\frac{w\sqrt{n}}{\mu}+1)\sqrt{n}w\gamma h] > 0$*, then the nonlinear system* (5.4) *is controllable.*

Proof. Let $X_1 = L^2([t_0,t_1];\mathbb{R}^{mn})$, $X_2 = L^2([t_0,t_1];\mathbb{R}^{n^2})$. By Lemma 5.4.4 the controllability of system (5.4) is equivalent to the solvability of the coupled system

$$\psi = \mathcal{M}_1 u,$$
$$u = u_1 - \mathcal{M}_2\psi,$$

where $\mathcal{M}_1 = (I-KN)^{-1}H : X_1 \to X_2$ and $\mathcal{M}_2 = RN : X_2 \to X_1$. By Theorem 5.4.11 the operator $(I-KN)^{-1}$ is continuous and satisfies the growth condition (5.40). Since the operator H is compact, it follows that operator $\mathcal{M}_1$ is also compact and satisfies the following growth condition

$$\|\mathcal{M}_1 u\| \leq \frac{hd\sqrt{n}}{\mu} + (\frac{w\sqrt{n}}{\mu}+1)h\|u\|.$$

Similarly it can be shown that $\mathcal{M}_2$ is continuous with growth condition

$$\begin{aligned}\|\mathcal{M}_2\psi\| &= \|R(N\psi)\| \\ &\leq \gamma\|G_1(\cdot,\psi(\cdot))\| \\ &\leq \gamma d\sqrt{n} + \gamma w\sqrt{n}\|\psi\|.\end{aligned}$$

Thus, the operator $\mathcal{M}_1$ and $\mathcal{M}_2$ satisfy all the conditions of Lemma 5.4.10, which implies the solvability of system (5.19) that in turn implies the controllability of the nonlinear system (5.4). □

We will finally give one example to illustrate our results.

Example 5.4.13. Consider the matrix Laypunov nonlinear differential equation

$$\begin{bmatrix} \dot{x}_{11}(t) & \dot{x}_{12}(t) \\ \dot{x}_{21}(t) & \dot{x}_{22}(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{bmatrix} + \begin{bmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix} + c \begin{bmatrix} \sin(x_{11}(t)) & \cos(x_{12}(t)) \\ \cos(x_{21}(t)) & \sin(x_{22}(t)) \end{bmatrix}, \tag{5.41}$$

By applying the Vec operator to above equation, we have the following equation of the form (5.4)

$$\begin{bmatrix} \dot{x}_{11}(t) \\ \dot{x}_{21}(t) \\ \dot{x}_{12}(t) \\ \dot{x}_{22}(t) \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 & 0 \\ 3 & 3 & 0 & 2 \\ 1 & 0 & 2 & 2 \\ 0 & 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \\ x_{12}(t) \\ x_{22}(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + c \begin{bmatrix} \sin(x_{11}(t)) \\ \cos(x_{21}(t)) \\ \cos(x_{12}(t)) \\ \sin(x_{22}(t)) \end{bmatrix}.$$

In this example,

$$A_1 = \begin{bmatrix} 2 & 2 & 2 & 0 \\ 3 & 3 & 0 & 2 \\ 1 & 0 & 2 & 2 \\ 0 & 1 & 3 & 3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix},$$

and the nonlinear operator G_1 is given by

$$G_1(t, x(t)) = c[\sin(x_{11}(t)), \cos(x_{21}(t)), \cos(x_{12}(t)), \sin(x_{22}(t))]^T,$$

where $x(t) = [x_{11}(t), x_{21}(t), x_{12}(t), x_{22}(t)]^T$.

In this example, we take $h(t,s) = (\sum e^{A_1 t})(\sum e^{-A_1 s})$. Clearly $\|\Phi(t,s)\| = \|e^{A_1(t-s)}\| \leq h(t,s)$. Let $c = \frac{1}{140000}$. Furthermore, it can be easily shown that

$$\left\| \begin{bmatrix} \sin(x_{11}(t)) - \sin(y_{11}(t)) & \cos(x_{12}(t)) - \cos(y_{12}(t)) \\ \cos(x_{21}(t)) - \cos(y_{21}(t)) & \sin(x_{22}(t)) - \sin(y_{22}(t)) \end{bmatrix} \right\| \leq \sqrt{2} \left\| \begin{bmatrix} x_{11}(t) - y_{11}(t) & x_{12}(t) - y_{12}(t) \\ x_{21}(t) - y_{21}(t) & x_{22}(t) - y_{22}(t) \end{bmatrix} \right\|.$$

The above equation shows that Lipschitz constant α is $\sqrt{2}$, which in turn implies that the value of Lipschitz constant β for nonlinear operator N is 2c ($\sqrt{2}\alpha$ c). Let the time interval $[t_0, t_1]$ be $[0, \cdot 1]$.

Using above definition of $h(t, s)$, the bounds for the norm of operators K, H and R are computed as 1.4390(k), 2.0351(h), and $3.3101 \times 10^4(\gamma)$, respectively. Then it can be easily shown that $k\beta < 1$ and $(\frac{\beta}{1-k\beta}\gamma h)(= .9623) < 1$ for sufficiently small value of c. Thus, all the conditions of Theorem 5.4.8 are satisfied. Hence the system (5.41) is completely controllable during time interval $[0, \cdot 1]$.

Remark 5.4.14. Note that sharper bounds for $\|K\|$, $\|H\|$ and $\|R\|$ can be obtained by suitably choosing the function $h(t, s)$. Thus, a higher value of c can be obtained.

Remark 5.4.15. The norm $\|R\|$ is proportional to the norm of the inverse of controllability Grammian $W^{-1}(t_0, t_1)$. Therefore value of c can be increased by decreasing the value of $\|W^{-1}(t_0, t_1)\|$.

5.5 Conclusion

In the chapter, we have studied controllability and established some sufficient conditions for the complete controllability of nonlinear MLS. In Murty et al. (2006), observability analysis for the linear MLS is done. We feel that our work can be extended to study the observability analysis of nonlinear MLS. Furthermore, the stabilizability of linear and nonlinear MLS can also be studied.

CHAPTER 6

Controllability of Impulsive Matrix Lyapunov Systems

6.1 Introduction

In this chapter, we investigate the complete controllability of the following matrix Lyapunov systems with impulse effects

$$\begin{cases} \dot{X}(t) = A(t)X(t) + X(t)B(t) + F(t)U(t) + G(t, X(t)), t \neq t_k, t \in [t_0, T] \\ X(t_k^+) = [I_n + D^k U(t_k)]X(t_k), k = 1, 2, \ldots, \rho \\ X(t_0) = X_0, \end{cases} \tag{6.1}$$

where the state $X(t)$ is an $n \times n$ real matrix, control $U(t)$ is an $m \times n$ real matrix. $A(t), B(t), F(t)$ are $n \times n$, $n \times n$, $n \times m$ real matrices with piecewise continuous entries and $t_0 \leq t_1 \leq t_2 \ldots \leq t_\rho \leq T$ are the time points at which impulse control $U(t_k)$ is given to the system. For each $k = 1, 2, \ldots, \rho$, $D^k U(t_k)$ is an $n \times n$ diagonal matrix such that $D^k U(t_k) = \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij}^k U_{ij}(t_k) I_n$, where I_n is the identity matrix on $\mathbb{R}^n$ and $d_{ij}^k \in \mathbb{R}$. $G(\cdot, \cdot) : \mathbb{R}^+ \times \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ is a nonlinear function and satisfies the 'Caratheodory conditions', that is, $G(\cdot, x)$ is measurable with respect to t for all $x \in \mathbb{R}^{n \times n}$ and $G(t, \cdot)$ is continuous with respect to x for almost all $t \in [t_0, T]$.

The control $u(t)$ is said to be impulsive if at $t = t_k, k = 1, 2, \ldots, \rho$, the pulses are regulated and chosen arbitrarily in rest of the domain. Study of such systems has received much attention in the literature due to the fact that many evolutionary processes, for instance, some motions of satellites, frequency modulated systems and bursting rhythm models in biology are impulsive in nature (cf. Lakshmikantham et al. (1989), Pandit and Deo (1982), Liu and Willms (1996)). Controllability of impulsive systems has been well investigated in the literature. For instance, in Guan et al. (2002a,b), the authors study the controllability and observability for a time-varying impulsive control systems and establish some sufficient and necessary conditions for the state controllability and state observability of the impulsive control systems. Furthermore, Xie and Wang (2004) have established necessary and sufficient conditions for the controllability of switched impulsive control systems.

Indeed, the controllability of many special cases of system (6.1) has been studied in the literature. For example, if $B(t) = 0$ and $G(t, x) = 0$ hold along with $D^k U(t_k) = 0$ for $k = 1, 2, \ldots, \rho$, then the system (6.1) reduces to linear time-varying control system whose controllability is well established in the literature, for example, Barnett and Cameron (1975), Zabczyk (2008). Leela et al. (1993) studied the controllability of a special case of system (6.1) with $B(t) = 0$, $G(t, x) = 0$, and $A(t)$, $F(t)$ are constant matrices. In George et al. (2000), complete controllability of system (6.1) with $B(t) = 0$ and $X(t) \in \mathbb{R}^n$ is investigated.

Recently Murty et al. (2006) studied the controllability of linear non-impulsive matrix Lypunov systems, that is, system (6.1) with $G = 0$ and $D^k U(t_k) = 0$ for $k = 1, 2, \ldots, \rho$. Furthermore, in Dubey and George (2013b) controllability of semi-linear non-impulsive matrix Lyapunov systems, that is, system (6.1) with $D^k U(t_k) = 0$ for $k = 1, 2, \ldots, \rho$, is established.

In the chapter, first we investigate the complete controllability of unperturbed (linear) system, that is, system (6.1) with $G = 0$. We then establish complete controllability of perturbed (nonlinear) system, that is, system (6.1) itself. The organization of the chapter is as follows:

In Section 6.2, we establish some sufficient conditions for the complete controllability of unperturbed system. Sufficient conditions for the complete controllability of perturbed system (6.1) are obtained in Section 6.3. The nonlinearities in the perturbed systems are assumed to be either Lipschitz type or monotone type. Finally, we conclude the chapter in Section 6.4.

6.2 Controllability of Linear Impulsive Matrix Lyapunov Systems

We will first consider the unperturbed (linear) IMLS, that is, the system (6.1) with $G = 0$. By applying Vec [1] operator to the unperturbed system, we have the following system equivalent to the unperturbed IMLS:

$$\begin{cases} \dot{\psi}(t) = A_1(t)\psi(t) + B_1(t)\hat{u}(t), t \neq t_k, t \in [t_0, T] \\ \psi(t_k^+) = [I_{n^2} + D^k\hat{u}(t_k)]\psi(t_k), k = 1, 2, \ldots, \rho \\ \psi(t_0) = \psi_0, \end{cases} \tag{6.2}$$

where $\psi(t) = Vec(X(t))$, $A_1(t) = (B^T(t) \otimes I_n) + (I_n \otimes A(t))$, $B_1(t) = I_n \otimes F(t)$, $\hat{u}(t) = Vec(U(t))$, $\psi_0 = Vec(X_0)$ and $D^k\hat{u}(t_k) = \sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}^k U_{ij}(t_k) I_{n^2}$.

The following lemma follows immediately from the Lemma 2·1 of Murty et al. (2006).

Lemma 6.2.1. *$\Phi_1(t,s)$ and $\Phi_2(t,s)$ are the transition matrices for systems $\dot{X}(t) = A(t)X(t)$ and $\dot{X}(t) = B^T(t)X(t)$, respectively. Then the transition matrix $\Phi(t,s)$ for the system $\dot{X}(t) = A_1(t)X(t)$ is given by*

$$\Phi(t,s) = \Phi_2(t,s) \otimes \Phi_1(t,s),$$

where $A_1(t) = (B^T(t) \otimes I_n) + (I_n \otimes A(t))$.

It can be shown that the solution of unperturbed (linear) system (6.2) in the time interval $[t_\rho, T]$ is given by

$$\begin{aligned} \psi(t) = & \sum_{i=1}^{\rho} \int_{t_{i-1}}^{t_i} \prod_{t_{i-1}<t_k<T} (I_{n^2} + D^k\hat{u}(t_k))\Phi(t,s)B_1(s)\hat{u}(s)ds \\ & + \int_{t_\rho}^{t} \Phi(t,s)B_1(s)\hat{u}(s)ds + \prod_{t_0<t_k<T} (I_{n^2} + D^k\hat{u}(t_k))\Phi(t,t_0)\psi_0, \end{aligned} \tag{6.3}$$

where $\Phi(t,\tau)$ is the transition matrix for the system $\dot{X}(t) = A_1(t)X(t)$.

[1] For the definition and properties of the "Vec" operator, refer Section 5.2 of Chapter 5.

Definition 6.2.2. *(Controllability) The nonlinear matrix Lyapunov system* (6.1) *is said to be controllable(completely controllable) on* $[t_0, T]$ *in the domain of controllability* $D \subset \mathbb{R}^{n\times n}$ *if for each pair of matrices* $X_0, X_1 \in D$*, there exists a control* $u \in L^2([t_0, T]; \mathbb{R}^{m\times n})$ *such that the solution of* (6.1) *together with* $X(t_0) = X_0$ *also satisfies* $X(T) = X_1$.

It can be easily shown that controllability of unperturbed matrix Lyapunov system is equivalent to the controllability of reduced impulsive linear system (6.2) (see Proposition 5.3.3 of Chapter 5).

We now introduce the following operators which we shall use throughout the chapter. Let $\mathrm{C} : L^2([t_\rho, T]; \mathbb{R}^{mn}) \to \mathbb{R}^{n^2}$ is defined by

$$\mathrm{C}u = \int_{t_\rho}^{T} \Phi(T, s)(I_n \otimes F(s))u(s)ds,$$

and $\mathrm{C}^* : \mathbb{R}^{n^2} \to L^2([t_\rho, T]; \mathbb{R}^{mn})$ is the adjoint of C and defined as follows

$$(\mathrm{C}^*\psi)(t) = (I_n \otimes F^T(t))\Phi^T(T, t)\psi.$$

The following lemma is crucial in establishing the controllability of the unperturbed system.

Lemma 6.2.3. *The unperturbed matrix Lyapunov system can be steered from any initial state* $X_0 \in \mathbb{R}^{n\times n}$ *to any desired state* $X_1 \in \mathbb{R}^{n\times n}$ *during time interval* $[t_0, T]$*, if* $Vec(X_1) \in R(\mathrm{C}) + span\{\Phi(T, t_0)Vec(X_0)\}$*, where* $R(\mathrm{C})$ *denotes the range of the operator* C.

Proof. Let $Vec(X_1) \in R(\mathrm{C}) + span\{\Phi(T, t_0)Vec(X_0)\}$. Then, there exists an $u(\cdot) \in L^2([t_\rho, T]; \mathbb{R}^{mn})$ and $\alpha \in \mathbb{R}$ such that

$$Vec(X_1) = \int_{t_\rho}^{T} \Phi(T, s)B_1(s)u(s)ds + \alpha\Phi(T, t_0)Vec(X_0). \tag{6.4}$$

Define a control $U(\cdot)$ in $L^2([t_0, T]; \mathbb{R}^{m\times n})$ as follows. First choose $U(t_k)$ for $k = 0, 1, 2, \ldots, \rho$, such that $\prod_{t_0<t_k<T} (I_{n^2} + D^k\hat{u}(t_k)) = \alpha I_{n^2}$ (such a choice is always pos-

sible) and in the rest of the domain let $\hat{u}(t) = Vec(U(t))$ can be given as follows:

$$\hat{u}(t) = \begin{cases} 0, \text{if } t \in \cup_{k=1}^{\rho}(t_{k-1}, t_k) \\ u(t), \text{if } t \in (t_\rho, T]. \end{cases} \tag{6.5}$$

By using Eq. (6.3), the solution of system (6.2) with the control $\hat{u}(\cdot)$ in the time interval $[t_\rho, T]$ is given by

$$\psi(t) = \alpha\Phi(t, t_0)\psi_0 + \int_{t_\rho}^{t} \Phi(t, s)B_1(s)\hat{u}(s)ds. \tag{6.6}$$

By substituting $t = T$ in the equation (6.6), we have

$$\psi(T) = Vec(X(T)) = \alpha\Phi(T, t_0)\psi_0 + \int_{t_\rho}^{T} \Phi(T, s)B_1(s)\hat{u}(s)ds. \tag{6.7}$$

Combining Eq. (6.4) and Eq. (6.7), we have $Vec(X(T)) = Vec(X_1)$ which in turn implies that $X(T) = X_1$. Thus, the control $U(\cdot)$ steers the unperturbed IMLS from the initial state X_0 to the state X_1 during time interval $[t_0, T]$. Hence the proof of the lemma follows. □

Remark 6.2.4. *The control $U(\cdot) \in L^2([t_0, T]; \mathbb{R}^{n\times m})$ that transfers the initial state $X_0 \in \mathbb{R}^{n\times n}$ to the desired target state $X_1 \in \mathbb{R}^{n\times n}$ is given by the following relation*

$$(Vec(U))(t) = \begin{cases} 0, \textit{if } t \in \cup_{k=1}^{\rho}(t_{k-1}, t_k) \\ B_1^T(t)\Phi^T(T, s)(t)W^{-1}(t_\rho, T)(Vec(X_1) - \\ \prod\limits_{t_0<t_k<T}(I_{n^2} + D^k\hat{u}(t_k)\Phi(T, t_0)Vec(X_0)), \textit{if } t \in (t_\rho, T]. \end{cases} \tag{6.8}$$

The following theorem gives the sufficient conditions for the controllability of the unperturbed system (6.2).

Theorem 6.2.5. *The unperturbed matrix Lyapunov system is completely controllable on time interval $[t_0, T]$ if any of the following conditions holds:*

(i) The operator C *is surjective.*

(ii) $W(t_\rho, T) = \int_{t_\rho}^{T} \Phi(T, \tau)B_1(\tau)B_1^T(\tau)\Phi^T(T, \tau)d\tau$ *is non-singular.*

Proof. Since we have shown in Lemma 6.2.3 that any desired state $X_1 \in \mathbb{R}^{n\times n}$ is attainable from any initial state $X_0 \in \mathbb{R}^{n\times n}$ if $Vec(X_1) \in R(\mathrm{C}) + span\{\Phi(T,t_0)Vec(X_0)\}$. Therefore the system will be completely controllable if

$$R(\mathrm{C}) + span\{\Phi(T,t_0)Vec(X_0)\} = \mathbb{R}^{n^2}.$$

And the above holds if and only if $R(\mathrm{C}) = \mathbb{R}^{n^2}$ or C is surjective. Hence the theorem follows under (i).

Since $R(\mathrm{C}) = R(\mathrm{CC}^*)$, where C^* is the adjoint of the operator C. Observe that $\mathrm{CC}^* = W(t_\rho, T)$, and invertibility of $W(t_\rho, T)$ is equivalent to the fact that CC^* is surjective and that holds if and only if C is surjective. Thus, (i) and (ii) are equivalent. Hence the proof of the theorem follows. □

Remark 6.2.6. *In general for the linear matrix Lyapunov systems the concept of complete controllability is equivalent to null controllability, that is, steering any state to* 0 *from any arbitrary initial state* X_0*. However, it is observed that for the impulsive linear matrix Lyapunov systems null controllability is much weaker than complete controllability. The impulsive linear matrix Lyapunov system is always null controllable without any condition due to the fact that* $0 \in R(\mathrm{C}) + span\{\Phi(T,t_0)X_0\}$.

Remark 6.2.7. *In case of* $A(t)$*,* $B(t)$ *and* $F(t)$ *are time invariant matrices, so are the* $A_1(t)$ *and* $B_1(t)$*. Then the unperturbed matrix Lyapunov system is completely controllable if*

$$rank[B_1|A_1B_1|A_1^2B_1|\dots|A_1^{n^2-1}B_1] = n^2.$$

Theorem 6.2.5 essentially states that the unperturbed system, that is system, (6.1) with $G = 0$ can be controlled to any state by suitable a control active only in the time interval $[t_\rho, T]$. We shall now show that it also possible to control the unperturbed system to any arbitrary state by some suitable control active in any of the time intervals $[t_{i-1}, t_i]$, for $i = 1, 2, \dots, n$.

Let the operator $\mathrm{C}_i : L^2([t_{i-1}, t_i]; \mathbb{R}^{mn}) \to \mathbb{R}^{n^2}$, $i = 1, 2 \dots, \rho$ be defined as follows:

$$\mathrm{C}_i u = \int_{t_{i-1}}^{t_i} \Phi(t_i, s)(I_n \otimes F(s))u(s)ds.$$

The following lemma establishes the controllability in terms of the operator C_i for $i = 1, 2 \ldots, n$.

Lemma 6.2.8. *The unperturbed matrix Lyapunov system can be steered from any initial state $X_0 \in \mathbb{R}^{n\times n}$ to any desired state $X_1 \in \mathbb{R}^{n\times n}$ during time interval $[t_0, T]$ if $\Phi(t_i, T)(\prod_{t_{i-1}<t_k<T}(I_{n^2} + D^k\hat{u}(t_k)))^{-1}Vec(X_1) \in R(\mathrm{C}_i) + span\{\Phi(t_i, t_0)Vec(X_0)\}$ for some $i \in \{1, 2, \ldots, \rho\}$, where $R(\mathrm{C}_i)$ denotes the range of the operator C_i.*

Proof. Let $\Phi(t_i, T)(\prod_{t_{i-1}<t_k<T}(I_{n^2}+D^k\hat{u}(t_k)))^{-1}Vec(X_1) \in R(\mathrm{C}_i)+span\{\Phi(t_i, t_0)Vec(X_0)\}$. Then, there exists an $u_i(\cdot) \in L^2([t_{i-1}, t_i]; \mathbb{R}^{mn})$ and $\alpha \in \mathbb{R}$ such that

$$Vec(X_1) = \left(\prod_{t_{i-1}<t_k<T}(I_{n^2} + D^k\hat{u}(t_k))\right)\Phi(T, t_i)\times \left(\int_{t_{i-1}}^{t_i}\Phi(t_i, s)B_1(s)u_i(s)ds + \alpha\Phi(t_i, t_0)Vec(X_0)\right). \quad (6.9)$$

Define a control $U(\cdot)$ in $L^2([t_0, T]; \mathbb{R}^{m\times n})$ as follows. First choose $U(t_k)$ for $k = 0, 1, 2, \ldots, \rho$, such that $\prod_{t_0<t_k<t_i}(I_{n^2} + D^k\hat{u}(t_k)) = \alpha I_{n^2}$ and in the rest of the domain let $\hat{u}(t) = Vec(U(t))$ be given as follows:

$$\hat{u}(t) = \begin{cases} 0, \text{if } t \in \cup_{k=1,k\neq i}^{\rho}(t_{k-1}, t_k) \cup (t_\rho, T] \\ u_i(t), \text{if } t \in (t_{i-1}, t_i). \end{cases} \quad (6.10)$$

By using Eq. (6.3), the solution of system (6.2) with the control $\hat{u}(\cdot)$ in the time interval $[t_\rho, T]$ is given by

$$\psi(t) = \left(\prod_{t_0<t_k<T}(I_{n^2} + D^k\hat{u}(t_k))\right)\Phi(t, t_0)\psi_0 + \int_{t_{i-1}}^{t_i}\left(\prod_{t_{i-1}<t_k<T}(I_{n^2} + D^k\hat{u}(t_k))\right)\Phi(t, s)B_1(s)u_i(s)ds. \quad (6.11)$$

From Eq. (6.11) it follows that

$$\psi(t) = \alpha \left(\prod_{t_{i-1}<t_k<T} (I_{n^2} + D^k \hat{u}(t_k)) \right) \Phi(t, t_0)\psi_0 + \int_{t_{i-1}}^{t_i} \left(\prod_{t_{i-1}<t_k<T} (I_{n^2} + D^k \hat{u}(t_k)) \right) \Phi(t, s) B_1(s) u_i(s) ds. \tag{6.12}$$

Evaluating the above expression at $t = T$ together with (6.9), we have

$$\begin{aligned} \psi(T) = Vec(X(T)) = \alpha(\prod_{t_{i-1}<t_k<T} (I_{n^2} + D^k \hat{u}(t_k)))\Phi(T, t_0)\psi_0 + \\ \int_{t_{i-1}}^{t_i} \left(\prod_{t_{i-1}<t_k<T} (I_{n^2} + D^k \hat{u}(t_k)) \right) \Phi(T, s) B_1(s) u_i(s) ds \\ = Vec(X_1). \end{aligned} \tag{6.13}$$

The last equation implies that $X(T) = X_1$. Thus, $U(\cdot)$ steers the system from the initial state X_0 to the state X_1 during time-interval $[t_0, T]$. Hence the proof of the lemma follows. □

Remark 6.2.9. *The control $U(\cdot) \in L^2([t_0, T]; \mathbb{R}^{n\times m})$ that transfers the initial state $X_0 \in \mathbb{R}^{n\times n}$ to the desired target state $X_1 \in \mathbb{R}^{n\times n}$ is given by the following relation*

$$(Vec(U))(t) = \begin{cases} 0, \text{ if } t \in \cup_{k=1,k\neq i}^{\rho} (t_{k-1}, t_k) \cup (t_\rho, T] \\ B_1^T(t)\Phi^T(t_i, s)(t) W^{-1}(t_{i-1}, t_i) \times \\ \big(\Phi(t_i, T) \left(\prod_{t_{i-1}<t_k<T} (I_{n^2} + D^k \hat{u}(t_k)) \right)^{-1} Vec(X_1) - \\ \prod_{t_0<t_k<t_i} (I_{n^2} + D^k \hat{u}(t_k)) \Phi(t_i, t_0) Vec(X_0)\big), \text{ if } t \in (t_\rho, T]. \end{cases} \tag{6.14}$$

The following theorem gives the sufficient conditions for the controllability of the unperturbed system (6.2) on terms of the operator C_i.

Theorem 6.2.10. *The unperturbed matrix Lyapunov system is completely controllable on time-interval $[t_0, T]$ if any of the following conditions holds:*

(i) The operator C_i is surjective.

(ii) $W(t_{i-1}, t_i) = \int_{t_{i-1}}^{t_i} \Phi(t_i, \tau) B_1(\tau) B_1^T(\tau) \Phi^T(t_i, \tau) d\tau$ *is non-singular.*

Proof. By Lemma 6.2.8 it follows that any desired state $X_1 \in \mathbb{R}^{n \times n}$ is attainable from any initial state $X_0 \in \mathbb{R}^{n \times n}$ if $\Phi(t_i, T)(\prod_{t_{i-1}<t_k<T} (I_{n^2} + D^k \hat{u}(t_k)))^{-1} Vec(X_1) \in R(\mathrm{C}_i) + span\{\Phi(t_i, t_0) Vec(X_0)\}$. Therefore the system will be completely controllable if

$$\prod_{t_{i-1}<t_k<T} (I_{n^2} + D^k \hat{u}(t_k)) \Phi(T, t_i)(R(\mathrm{C}_i) + span\{\Phi(t_i, t_0) Vec(X_0)\}) = \mathbb{R}^{n^2}.$$

Since $\prod_{t_{i-1}<t_k<T} (I_{n^2} + D^k \hat{u}(t_k)) \Phi(T, t_i)$ is a non-singular matrix therefore the above holds if and only if $R(\mathrm{C}_i) = \mathbb{R}^{n^2}$ or C_i is surjective.

Further, since $R(\mathrm{C}_i) = R(\mathrm{C}_i \mathrm{C}_i^*)$, where C_i^* is the adjoint of the operator C_i. Observe that $\mathrm{C}_i \mathrm{C}_i^* = W(t_{i-1}, t_i)$, and invertibility of $W(t_{i-1}, t_i)$ is equivalent to the fact that $\mathrm{C}_i \mathrm{C}_i^*$ is surjective and that holds if and only if C_i is surjective. Thus, (i) and (ii) are equivalent. Hence the proof of theorem follows. □

6.3 Controllability of Semilinear Impulsive Matrix Lyapunov Systems

We will now provide sufficient conditions for the complete controllability of the perturbed system (6.1). By applying Vec operator to the equation (6.1), equation (6.1) can be equivalently written as follows:

$$\begin{cases} \dot{\psi}(t) = A_1(t)\psi(t) + B_1(t)\hat{u}(t) + G_1(t, \psi(t)), t \neq t_k, t \in [t_0, T] \\ \psi(t_k^+) = [I_{n^2} + D^k \hat{u}(t_k)]\psi(t_k), k = 1, 2, \ldots, \rho \\ \psi(t_0) = \psi_0, \end{cases} \tag{6.15}$$

where $\psi(t) = Vec(X(t))$, $A_1(t) = (B^T(t) \otimes I_n) + (I_n \otimes A(t))$, $B_1(t) = I_n \otimes F(t)$, $\hat{u}(t) = Vec(U(t))$, $G_1(t, \psi(t)) = Vec(G(t, X(t)))$, $\psi_0 = Vec(X_0)$ and $D^k \hat{u}(t_k) = \sum_{i=1}^m \sum_{j=1}^n d_{ij}^k u_{ij}(t_k) I_{n^2}$. Thus, the controllability of the system (6.1) is equivalent to the controllability of the system (6.15).

The solution of (6.15) in the time interval $[t_\rho, T]$ satisfies

$$\psi(t) = \tilde{\psi_0} + \int_{t_\rho}^{t} \Phi(t,s)(I_n \otimes F(s))\hat{u}(s)ds + \int_{t_\rho}^{t} \Phi(t,s)G_1(s,\psi(s))ds, \tag{6.16}$$

where $\tilde{\psi_0}$ is given by

$$\begin{aligned}\tilde{\psi_0} = &\prod_{t_0<t_k<T}(I_{n^2} + D^k\hat{u}(t_k))\Phi(t,t_0)\psi_0 + \\ &\sum_{i=1}^{\rho}\int_{t_{i-1}}^{t_i}\prod_{t_{i-1}<t_k<T}(I_{n^2} + D^k\hat{u}(t_k))\Phi(t,s)B_1(s)\hat{u}(s)ds + \\ &\sum_{i=1}^{\rho}\int_{t_{i-1}}^{t_i}\prod_{t_{i-1}<t_k<T}(I_{n^2} + D^k\hat{u}(t_k))\Phi(t,s)G_1(s,\psi(s))ds.\end{aligned} \tag{6.17}$$

Since we are looking for sufficient conditions for the complete controllability of the (6.15). Let us first choose the control $U(\cdot)$ in $L^2([t_0,T];\mathbb{R}^{m\times n})$ such that

$$I_{n^2} + D^k\hat{u}(t_k) = 0, k = 1,2,\dots\rho.$$

With such a choice of $\hat{u}(\cdot)$, (6.16) reduces to

$$\psi(t) = \int_{t_\rho}^{t} \Phi(t,s)B_1(s)\hat{u}(s)ds + \int_{t_\rho}^{t} \Phi(t,s)G_1(s,\psi(s))ds. \tag{6.18}$$

Suppose further that the control $\hat{u}(\cdot)$ is such that the Volterra-type integral equation (6.18) also satisfies $\psi(T) = \psi_1$. That is,

$$\psi_1 = \int_{t_\rho}^{T} \Phi(T,s)B_1(s)\hat{u}(s)ds + \int_{t_\rho}^{T} \Phi(T,s)G_1(s,\psi(s))ds. \tag{6.19}$$

By rearrangements of the terms in (6.19)

$$\psi_1 - \int_{t_\rho}^{T} \Phi(T,s)G_1(s,\psi(s))ds = \int_{t_\rho}^{T} \Phi(T,s)B_1(s)\hat{u}(s)ds. \tag{6.20}$$

A suitable choice of $\hat{u}(t)$ in time interval $[t_\rho, T]$ that satisfies (6.20) can be given as follows:

$$\hat{u}(t) = \mathrm{C}^*((\mathrm{CC}^*)^{-1}[\psi_1 - \int_{t_\rho}^{T} \Phi(T,s)G_1(s,\psi(s))ds])(t). \tag{6.21}$$

This shows that the complete controllability of the system (6.15) is equivalent to the solvability of (6.18) and (6.21), together. Now we will discuss the solvability of the coupled equations (6.18) and (6.21). Let us introduce following operators.

Let $X_1 = L^2([t_\rho, T]; \mathbb{R}^{mn})$, $X_2 = L^2([t_\rho, T]; \mathbb{R}^{n^2})$. Define operators $K, N : X_2 \to X_2$, $H : X_1 \to X_2$ and $R : X_2 \to X_1$ as follows:

$$(K\psi)(t) = \int_{t_\rho}^{t} \Phi(t,\tau)\psi(\tau)d\tau, \quad (N\psi)(t) = G_1(t, \psi(t)),$$
$$(Hu)(t) = \int_{t_\rho}^{t} \Phi(t,\tau)B_1(\tau)u(\tau)d\tau,$$
$$(R\psi)(t) = \mathrm{C}^*((\mathrm{C}\mathrm{C}^*)^{-1} \int_{t_\rho}^{T} \Phi(T,\tau)\psi(\tau)d\tau)(t).$$

Using above operators equations (6.18) and (6.21) can be written as a pair of operator equations

$$\begin{aligned} \psi &= KN\psi + Hu \\ u &= u_1 - RN\psi, \end{aligned} \tag{6.22}$$

where $u_1(t) = \mathrm{C}^*((\mathrm{C}\mathrm{C}^*)^{-1}\psi_1)(t)$. Thus, based on the above discussion we have the following important result.

Theorem 6.3.1. *The nonlinear system* (6.1) *is completely controllable if the coupled system* (6.22) *is uniquely globally solvable.*

Proof. If the pair (6.22) is uniquely globally solvable, then the control given by $u(t) = (u_1 - RN\psi)(t)$ for $t \in [t_\rho, T]$ exists and well defined, moreover by substituting control $u(t)$ in the equation $\psi = KN\psi + Hu$, we have $\psi(T) = \psi_1$. Now define a control $\hat{u}(\cdot)$ in $L^2([t_0, T], \mathbb{R}^{mn})$ as follows:

$$\hat{u}(t) = \begin{cases} v(t), & t \notin [t_\rho, T], t \neq t_k, k = 1, 2, \dots, \rho \\ u_1 - RN\psi, & t \in [t_\rho, T], \end{cases} \tag{6.23}$$

and $\hat{u}(t_k)$ are chosen so that $I_{n^2} + D^k\hat{u}(t_k) = 0$, for each $k = 1, 2, \dots, \rho$, and $v(\cdot)$ is any arbitrary function in $L^2([t_0, t_\rho], \mathbb{R}^{mn})$. Then it can be easily shown that the solution of (6.15) with the control $\hat{u}$ satisfies $\psi(T) = \psi_1$. That is, the system (6.15) is completely controllable and so does (6.1). □

Now we will give sufficient conditions for the solvability of the coupled system (6.22).

6.3.1 Controllability Results under Lipschitzian Nonlinearities

Let us now make the following assumptions:

(A1) Let $b = \sup_{t_\rho \le t \le T} \|B_1(t)\|$ and the transition matrix $\Phi(t,s)$ is such that $\|\Phi(t,s)\| \le h(t,s)$, where $h(\cdot,\cdot) : [t_\rho, T] \times [t_\rho, T] \to \mathbb{R}^+$ is a function satisfying

$$\Big[\int_{t_\rho}^{T} \int_{t_\rho}^{t} h^2(t,s)\, ds\, dt\Big]^{\frac{1}{2}} = k < \infty.$$

(A2) The function $G : [t_0, T] \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ satisfies the 'Caratheodory conditions'. Further, G satisfies Lipschitz condition in the time interval $[t_\rho, T]$ with Lipschitz constant α. That is, for $t \in [t_\rho, T]$ and $x, y \in \mathbb{R}^{n\times n}$,

$$\|G(t,x) - G(t,y)\| \le \alpha \|x - y\|.$$

Lemma 6.3.2. *Under Assumptions* (A1)–(A2)*, the bounds for* $\|K\|$*,* $\|H\|$ *and* $\|R\|$ *are estimated as* $\|K\| \le k$*,* $\|H\| \le bk \triangleq h$ *and* $\|R\| \le bk_1^2 c \triangleq \gamma$*, where* $c = \|(CC^*)^{-1}\|$ *and* $k_1 = [\int_{t_\rho}^{T} h^2(T,s) ds]^{\frac{1}{2}}$*. Furthermore, the nonlinear operator* N *is Lipschitz continuous and bounded from* X_2 *into itself with Lipschitz constant* $\beta = \sqrt{n}\alpha$*.*

Proof. The proof follows from Lemma 5.4.5 and Lemma (5.4.6) of Chapter 5. □

Theorem 6.3.3. *Let* β *be the Lipschitz constant for the nonlinear operator* N*. Suppose that controllability Grammian* $W(t_\rho, T)$ *or* CC* *is nonsingular and Assumptions* (A1)–(A2) *are satisfied with* $k\beta < 1$ *and* $(\frac{\beta}{1-k\beta})\gamma h < 1$ *then we have:*

(i) Coupled system (6.22) *is uniquely globally solvable.*

(ii) The control matrix $U(t) \in \mathbb{R}^{m\times n}$ *steering the system* (6.1) *from zero initial state at time* t_0 *to a desired state* X_1 *at time* T *can be approximated during time interval*

$[t_\rho, T]$ by the iterates $U^n(t)$ defined by

$$Vec(U^{(n)})(t) = (\mathrm{C}^*(\mathrm{CC}^*)^{-1}[Vec(X_1) - \int_{t_\rho}^{T} \Phi(T,s)G_1(s, Vec(X^{(n)})(s))ds])(t). \tag{6.24}$$

The state matrix approximation $X^{(n+1)}$ at $(n+1)^{th}$ stage during the time interval $[t_\rho, T]$ is given by the iterates

$$Vec(X_{j+1}^{(n+1)})(t) = \int_{t_\rho}^{t} \Phi(t,s)G_1(s, Vec(X_j^{(n+1)})(s))ds + \int_{t_\rho}^{t} \Phi(t,s)B_1(s)Vec(U^{(n)})(s)ds. \tag{6.25}$$

Proof. By using Theorem 5.4.8 of Chapter 5, it can be easily shown that the system (6.22) is uniquely globally solvable. Furthermore, approximation schemes for the control and state matrices can be easily obtained by closely following the proof of Theorem 5.4.8. □

The conditions of the Theorem 6.3.3 are more general and wide in nature. In particular, they can be satisfied if one chooses T and t_ρ sufficiently close to each other. However, in case when T and t_ρ are sufficiently close to each other, we establish the controllability of (6.1) independently without using Theorem 6.3.3. We will first define the solution operator.

Definition 6.3.4. *The solution operator $S : L^2([t_\rho, T]; \mathbb{R}^{mn}) \to L^2([t_\rho, T]; \mathbb{R}^{n^2})$ is defined by $Su = \psi$, where ψ satisfies the following:*

$$\psi(t) = \int_{t_\rho}^{t} \Phi(t,s)B_1(s)u(s)ds + \int_{t_\rho}^{t} \Phi(t,s)G_1(s, \psi(s))ds. \tag{6.26}$$

There are many conditions under which the solution operator is well defined and continuous (cf. George (1995), Joshi and George (1989)). The following lemma states the properties of the solution operator that we will use in our main result.

Lemma 6.3.5. *Suppose that Assumptions* (A1)–(A2) *hold along with the additional condition $k\alpha\sqrt{n} < 1$, then the solution operator S is well defined and Lipschitz continuous with Lipschitz constant $\frac{kb}{1-k\alpha\sqrt{n}}$.*

Proof. By using operators K and N, Eq. (6.26) gives rise to the following entity:

$$Su = KP_u + KNSu, \tag{6.27}$$

where $P_u \in L^2([t_\rho, T]; \mathbb{R}^{n^2})$ is defined by $P_u(t) = B_1(t)u(t)$. From Eq. (6.27), it follows that

$$Su = (I_{n^2} - KN)^{-1}KP_u. \tag{6.28}$$

For given any $u, v \in L^2([t_\rho, T]; \mathbb{R}^{mn})$, Eq. (6.28) implies

$$\|Su - Sv\| = \|(I_{n^2} - KN)^{-1}KP_u - (I_{n^2} - KN)^{-1}KP_v\|.$$

Now, by using Theorem 5.4.7 of Chapter 5, it follows that the operator $(I_{n^2} - KN)^{-1}$ is Lipschitz continuous with Lipschitz constant $\frac{1}{1-k\alpha\sqrt{n}}$. Thus, we have,

$$\|Su - Sv\| \leq \frac{1}{1 - k\alpha\sqrt{n}}\|K(P_u - P_v)\| \tag{6.29}$$

$$\leq \frac{1}{1 - k\alpha\sqrt{n}}kb\|u - v\|. \tag{6.30}$$

The last equation implies that S is Lipschitz continuous with Lipschitz constant $\frac{1}{1-k\alpha\sqrt{n}}kb$. By taking $u = v$, it also follows from the last equation that $Su = Sv$; this shows that S is well defined. Hence the lemma. □

Theorem 6.3.6. *Suppose the following conditions hold:*

(i) $W(t_\rho, T)$ *is non-singular.*

(ii) Assumptions (A1)–(A2) *hold.*

(iii) t_ρ *and* T *are sufficiently close.*

Then the system (6.1) *is completely controllable.*

Proof. By using (ii), (iii) and Lemma 6.3.5, it can be shown that the solution operator S is well defined and Lipschitz continuous. Now, the complete controllability of (6.1) follows from the solvability of the equation

$$\psi_1 = \psi(T) = \int_{t_\rho}^{T} \Phi(T,s)B_1(s)u(s)ds + \int_{t_\rho}^{T} \Phi(T,s)G_1(s,(Su)(s))ds. \tag{6.31}$$

By replacing u by $\mathrm{C}^*(W^{-1}(t_\rho,T)v)$ in (6.31), we have

$$v = \psi_1 + N_1 v, \tag{6.32}$$

where $N_1 : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ is a nonlinear operator defined by

$$N_1 v = -\int_{t_\rho}^{T} \Phi(T,s)G_1(s,(S\mathrm{C}^*(W^{-1}(t_\rho,T)v))(s))ds.$$

By using Lemma 6.3.5 and Lipschitz continuity of G_1, it can be shown that N_1 is Lipschitz continuous. Furthermore, by invoking condition (iii), it follows that N_1 is a contraction. Thus, by the Banach contraction principle (6.32) has a unique solution. Hence the proof of theorem follows. □

Remark 6.3.7. *The control $u(t)$ in time interval $[t_\rho, T]$ is given by $u(t) = \mathrm{C}^*(W^{-1}(t_\rho,T)v)(t)$, where v is the unique limit of the iterates $\{v_n\}$, given by*

$$v_{n+1} = \psi_1 + N_1 v_n$$

starting from any $v_0 \in \mathbb{R}^{n^2}$.

6.3.2 Controllability Results under Monotone Nonlinearities

Now we shall give conditions for the controllability of the system (6.1) when the nonlinear term G is not Lipschitz continuous. However, we assume G satisfies monotone type of conditions. Let us make following assumptions:

(B1) There exists a positive constant μ such that the matrix $A_1(t)$ satisfies

$$< -A_1(t)\psi, \psi > \geq \mu\|\psi\|^2 \text{ for } t \in [t_\rho, T].$$

(B2) The nonlinear function $-G$ is monotone. In fact $-G$ satisfies a weaker condition than monotonicity as given below. Given any $x_1, x_2 \in \mathbb{R}^{n\times n}$

$$< (G(t,x_1) - G(t,x_2))e_j, (x_1 - x_2)e_j > \leq 0, 1 \leq j \leq n,$$

where $\{e_j\}$ denotes the canonical basis in $\mathbb{R}^n$ and $t \in [t_\rho, T]$.

(B3) G also satisfies a growth condition of the form

$$\|G(t,x)\| \leq d(t) + w\|x\|,$$

for all $(t,x) \in [t_\rho, T] \times \mathbb{R}^{n\times n}$, $d(\cdot) \in L^2([t_\rho, T]; \mathbb{R})$ and $w > 0$.

The following theorem guarantees the controllability of impulsive matrix Lyapunov systems with monotone nonlinearities satisfying assumptions $(B2)$ and $(B3)$. Before establishing the results, we will give examples of monotone but not Lipschitzian nonlinearities satisfying assumptions $(B2)$ and $(B3)$. Let $G(t,x) : [t_\rho, T] \times \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$G(t,x) = \begin{cases} 0 & \text{if } x \in [-1,1] \\ -|x+1|^{\frac{1}{2}} & \text{if } x \in (-\infty,-1] \\ |x-1|^{\frac{1}{2}} & \text{if } x \in (1,\infty]. \end{cases}$$

Clearly the function defined above is monotonically increasing and bounded by linear growth constant $w = 1$. The function G is not globally Lipschitz as the derivative of G at points 1 and -1 are unbounded. Many other similar examples can be obtained.

Theorem 6.3.8. *Suppose that the linear system* (6.2) *is controllable and the Assumptions* (A1) *and* (B1)–(B3) *are satisfied. If* $[1-(\frac{w\sqrt{n}}{\mu}+1)\sqrt{n}w\gamma h] > 0$*, then the nonlinear system* (6.1) *is controllable.*

Proof. Under the assumptions of the theorem, it follows by using the results of Theorem 5.4.12 that the coupled system (6.22) is solvable which in turn implies that the system (6.1) is controllable. □

We will now provide an example to demonstrate the controllability of impulsive systems.

Example 6.3.9. *Consider the following nonlinear matrix Laypunov system with impulse effect*

$$\begin{aligned}\begin{bmatrix}\dot{x}_{11}(t) & \dot{x}_{12}(t)\\ \dot{x}_{21}(t) & \dot{x}_{22}(t)\end{bmatrix} = \begin{bmatrix}1 & 1\\ 3 & 2\end{bmatrix}\begin{bmatrix}x_{11}(t) & x_{12}(t)\\ x_{21}(t) & x_{22}(t)\end{bmatrix} + \begin{bmatrix}x_{11}(t) & x_{12}(t)\\ x_{21}(t) & x_{22}(t)\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\\ + \begin{bmatrix}1\\ 1\end{bmatrix}\begin{bmatrix}u_1(t) & u_2(t)\end{bmatrix} + c\begin{bmatrix}\sin(x_{11}(t)) & \cos(x_{12}(t))\\ \cos(x_{21}(t)) & \sin(x_{22}(t))\end{bmatrix},\end{aligned} \tag{6.33}$$

By applying the Vec *operator to above equation, we have the following equation of the form* (6.15)

$$\begin{bmatrix}\dot{x}_{11}(t)\\ \dot{x}_{21}(t)\\ \dot{x}_{12}(t)\\ \dot{x}_{22}(t)\end{bmatrix} = \begin{bmatrix}2 & 1 & 0 & 0\\ 3 & 3 & 0 & 0\\ 1 & 0 & 2 & 1\\ 0 & 1 & 3 & 3\end{bmatrix}\begin{bmatrix}x_{11}(t)\\ x_{21}(t)\\ x_{12}(t)\\ x_{22}(t)\end{bmatrix} + \begin{bmatrix}1 & 0\\ 1 & 0\\ 0 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix} + c\begin{bmatrix}\sin(x_{11}(t))\\ \cos(x_{21}(t))\\ \cos(x_{12}(t))\\ \sin(x_{22}(t))\end{bmatrix}.$$

Let the time interval under consideration is $[0, 1]$ *and the impulse point are taken as* $\cdot 3 \leq \cdot 6 \leq \cdot 95$. *Let*

$$X(\cdot 3^+) = [I_2 + 2U_{11}(\cdot 3)I_2 + 3U_{12}(\cdot 3)I_2]X(\cdot 3), \tag{6.34}$$

$$X(\cdot 6^+) = [I_2 + U_{11}(\cdot 6)I_2 + 4U_{12}(\cdot 6)I_2]X(\cdot 6), \tag{6.35}$$

$$X(\cdot 95^+) = [I_2 + U_{11}(\cdot 95)I_2 + 5U_{12}(\cdot 95)I_2]X(\cdot 95). \tag{6.36}$$

In the setting of above example we have taken $h(t, s) = (\sum e^{A_1 t})(\sum e^{-A_1 s})$. $t_\rho = \cdot 95$ *and* $T = 1$. *The value of* c *is* $\frac{1}{17000}$. *The value of Lipschitz constant* β *for the nonlinear operator* N *is* $\frac{2}{17000}$. *The bounds for the norms of the operators* K, H *and* R *are computed as* $.3616$*(k),* $.5113$*(h), and* $1.6379 \times 10^4(\gamma)$*, respectively. Thus, the constants* $k\beta < 1$ *and* $(\frac{\beta}{1-k\beta})\gamma h = .9853(< 1)$. *Hence in accordance with Theorem 6.3.3 the system* (6.33) *is completely controllable. The values of the control at impulse points can be taken as* $U(\cdot 3) = (1, -1)$, $U(\cdot 6) = (1, -\cdot 5)$ *and* $U(\cdot 95) = (4, -1)$, *and in the time interval* $[\cdot 95, 1]$ *the control* $U(t)$ *will be synthesized by using the iterates in*

Theorem 6.3.3.

6.4 Conclusion

In the chapter, we have established some sufficient conditions for the complete controllability of linear and nonlinear impulsive matrix Lyapunov systems. For the linear IMLS, we have shown that if $W(t_{i-1}, t_i)$ for any $1 \leq i \leq \rho$ or $W(t_\rho, T)$ is invertible then the linear IMLS is completely controllable. For the nonlinear IMLS, we have shown that if $W(t_\rho, T)$ is invertible and t_ρ and T are sufficiently close, then with smooth nonlinearities, (for instance, Lipschitzian nonlinearities) the nonlinear IMLS is completely controllable. For the future work in this direction, we feel that controllability of the nonlinear impulsive matrix Lyapunov systems can be established by using the surjectivity of the operators C_i for any $1 \leq i \leq \rho$.

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APPENDIX A

Matlab Simulation Scripts

A.1 M-scripts for Evolution of Solutions of Fuzzy Dynamical Systems

The following m-script is used in simulating the Example 3.2.6 in Chapter 3. We have used the results of Theorem 3.2.1 in this script.

```
clc
clear all;
close all;
options = odeset('RelTol',1e-4,'AbsTol',
          [1e-6 1e-6 1e-6 1e-6]);
s = sym('s');
p = 0;
simTime = 1
% Storage variables for propagated membership functions
x1,y1,y2 = [];
% Storage variables for Input grade functions
x3,y3,y4 = [];
out3 = sparse(100,5);
in   = sparse(100,5);
for i=1 : 100
    p=p+.01;
    input = subs([s s/2 2-s 1-(s/2)],p);
    [T,Y] = ode45(@ NonLinFuzzyPropwithInput(t,y,p),
              [0 simTime], input, options);
% This if condition is used to capture the plot for
```

```
% various $\alpha$-sections after every .2 second.
      if(mod(i,20) == 0)
         figure
         plot(T, Y(:,1),'r', T,Y(:,3),'b')
         title('Membership grade for x1')
         figure
         plot(T, Y(:,2),'r',T,Y(:,4),'b')
         title('Membership grade for x2')
      end
   n =  size(T,1);
   out3(i,:) = [p Y(n,1) Y(n,2) Y(n,3) Y(n,4)];
   in(i,:)  =  [p Y(1,1) Y(1,2) Y(1,3) Y(1,4)];
end
% Populate data for plots of initial membership functions.
for i=1 :100
    x3([2*i-1 2*i],1) = [in(i,1), in(i,1)]';
    y3([2*i-1 2*i],1) = [in(i,2), in(i,4)]';
    y4([2*i-1 2*i],1) = [in(i,3), in(i,5)]';
end
figure
% Plot of input membership function $\mu_{1}$
% for variable $x_1$.
plot(y3,x3,'.')
hold on;
% Plot of input membership function $\mu_{2}$
% for variable $x_2$.
plot(y4,x3,'.')
ylim([0 1]);
hold off
pause
% Populate data for plots of propagated membership
% functions at time simTime =1.
```

```
for i=1 :100
   x1([2*i-1 2*i],1) = [out3(i,1), out3(i,1)]';
   y1([2*i-1 2*i],1) = [out3(i,2), out3(i,4)]';
   y2([2*i-1 2*i],1) = [out3(i,3), out3(i,5)]';
end
figure
% Plot for propagated membership function for $x_1$
% at time t = 1.
plot(y1,x1,'.')
hold on;
% Plot for propagated membership function for $x_2$
% at time t = 1.
plot(y2,x1,'.')
ylim([0,1])
hold off
```

The function file $NonLinFuzzyPropwithInput.m$ contains the following $m-$ scripts.

```
function dy = NonLinFuzzyPropwithInput(t,y,a)
delta = sqrt(1-(1/log(exp(1)/a)));
dy = zeros(4,1);
dy(1) = -y(4) * y(4) + cos(t)+(t-1+a);
dy(2) = -y(3) * y(3) + sin(t)+(t-delta);
dy(3) = -y(2) * y(2) + cos(t)+(t+1-a);
dy(4) = -y(1) * y(1) + sin(t)+(t+delta);
```

The following m-script is used in Example 3.3.6 in Chapter 3.

```
clear all;
clc;
x1 = -1:.0005:1;
y1 = exp(1).*exp(-(1./(1-(x1.^2))));
x2 = -.5:.0005:.5;
```

```
y2 = exp(1).*exp(-((1/4)./((1/4)-(x2.^2))));
% Plot of membership function for variable x_1.
plot(x1,y1)
hold
% Plot of membership function for variable x_2.
plot(x2,y2)
hold off
ylim([0 1]);
% Transition matrix evaluated at t=1 for the system
% used in Example 3.3.6
A = [1 1 0 0; 0 1 0 0; 0 0 1 1; 0 0 0 1]
a = sym('a');
s = sym('s');
B = [s 0; 0 -s; 0 s; -s 0]
% Initial condition
x0 = [-(1-1/(log(exp(1)/a)))^.5
       -((1-1/(log(exp(1)/a)))^.5)/2
       (1-1/(log(exp(1)/a)))^.5
       ((1-1/(log(exp(1)/a)))^.5)/2]';
incre = .01;
p= .005;
% $\alpha$-level set for the fuzzy control function u(s)
u = [a+s-1 s+1-a]';
x = (expm(A*1))*x0 +
    int(expm(A-[s^2 s 0 0; 0 s^2 0 0;
    0 0 s^2 s; 0 0 0 s^2])*B*u,s,0,1);
for i=1 : 100
    x_1 = x([1 3],:);
    x_2 = x([2 4],:);
    level = subs(x_1,p);
    level_1 =  subs(x_2,p);
    x1_a([2*i-1 2*i],1) = level;
```

```
    x2_a([2*i-1 2*i],1) = level_1;
    y1_a([2*i-1 2*i],1) = [p p]';
    y2_a([2*i-1 2*i],1) = [p p]';
    p = p+incre;
end
figure
% Plot of the propagated membership function for
% variable x_1 at t=1
plot(x1_a,y1_a,'.');
hold on
% Plot of the propagated membership function for
% variable x_2 at t=1
plot(x2_a,y2_a,'.');
ylim([0 1]);
hold off
```

A.2 M-scripts for Controllability of Fuzzy Dynamical Systems

The following m-scripts is used to simulate the Example 4.2.16 in the Chapter 4. Results of Theorem 4.2.5 are used in this script.

```
clear all;
close all;
clc;
% Initial membership function.
 x1 = 0:.01:1;
 y1(1,1:51)   = 2.*x1(1,1:51);
 y1(1,52:101) = 2- 2.*x1(1,52:101);
 figure
 plot(x1,y1);
```

```
% Target membership function at final time.
x1 = 0:.01:8;
y1(1,1:401)   = .25.*x1(1,1:401);
y1(1,402:801) = 2- .25.*x1(1,402:801);
figure
plot(x1,y1);
% Final time
T = 1;
% Matrices obtained after the flip operations on the
% original system.
A = [0 -1;-1 0];
B = [0 -2;-2 0];
% Declare some symbolic variables.
s = sym('s','real');
a = sym('a','real');
t = sym('t','real');
% $\alpha$-level sets for initial and
% target membership functions.
x0 = [a/2;1-(a/2)];
x1 = [4*a; 8-4*a];
% Computation of controllability grammian and controller
% that steers initial level set to target level set.
W = int((expm(A*(-t))*B*B'*expm(A'*(-t))),t,0,T);
u = B'*expm(A'*(-t))*inv(W)*(expm(A*(-T))*x1 - x0);
U = subs(u,s);
% Change the value of t to compute state components
% at various instances. Default value for t is 1.
t=1
% State vector computation
x =(expm(A*t))*x0 + int(expm(A*(t-s))*B*U,s,0,t);
incre = .01;   p= .01;
% Generating propagated fuzzy state
```

```
for i=1 : 100
    level = subs(x ,p);
    xdata([2*i-1 2*i],1) = level;
    ydata([2*i-1 2*i],1) = [p p]';
    p = p+incre;
end
figure
plot(xdata,ydata,'.');
ylim([0 1]); xlim([0 9])
set(gca,'XTick',[1:9]);
t=sym('t')
% Plot of state and control function at $\alpha$ = .5"
x0 = [.25;.75];
x1 = [2; 6];
u = B'*expm(A'*(-t))*inv(W)*(expm(A*(-T))*x1 - x0);
U = subs(u,s);
x =(expm(A*t))*x0 + int(expm(A*(t-s))*B*U,s,0,t);
tdata = linspace(0,1,200)
odata = zeros(2,200)
udata = zeros(2,200)
for i=1 : size(tdata,2)
    odata(:,i) = subs(x,tdata(i));
    udata(:,i) = subs(u,tdata(i));
end
% Plot for the lower and upper cuts of the states
figure
plot(tdata,odata(1,:),'.',tdata,odata(2,:),'.')
% Plot for the lower and upper cuts of the control
figure
plot(tdata,udata(1,:),'.',tdata,udata(2,:),'.')
```

In Example 4.2.15 of Chapter 4, the following m-script is employed. Results of the Theorem 4.2.12 are used to compute the fuzzy steering control.

```
clear all;
clc;
% Initial membership function at t=0.
x1 = -.5:.0005:.5;
y1 = exp(1).*exp(-((1/4)./((1/4)-(x1.^2))));
plot(x1,y1);
% Target membership function t=1
x1 = -2:.0005:2;
y1 = exp(1).*exp(-(1./(1-((x1/2).$\^$ 2))));
plot(x1,y1);
% Final time
T = 1;
% Matrices obtained after the flip operations.
A = [1 0;0 1];
B = [1 0;0 1];
s = sym('s','real');
t = sym('t','real');
a = sym('a','real');
% alpha-cuts of initial membership functions
x0 = [-(1/2)*sqrt(1-(1/log(exp(1)/a)));
           (1/2)*sqrt(1-(1/log(exp(1)/a)))];
x0_rev = [(1/2)*sqrt(1-(1/log(exp(1)/a)));
             -(1/2)*sqrt(1-(1/log(exp(1)/a)))];
% Some metadata variables for calculations
 A1 = 1; C  = expm(A1*(1-0));
 z  = sqrt(1-(1/log(exp(1)/a)));
 p1 = -2*sqrt(1-(1/log(exp(1)/a)))+ C*z;
 p2 = 2*sqrt(1-(1/log(exp(1)/a))) - C*z;
% alpha level sets for the  $\widetilde{X_1}$
```

```
x1 = [p1;p2]
% Controllability Grammian and computation of steering control.
W = int((expm(A*(-t))*B*B'*expm(A'*(-t))),t,0,T);
u = B'*expm(A'*(-t))*inv(W)*(expm(A*(-T))*x1 - x0_rev);
U = subs(u,s);
% Change the value of $t \in [0,T]$ to get system state
% at desired time instant. Default value for t is 1 .
t=1
% State vector computation
x =(expm(A*t))*x0 + int(expm(A*(t-s))*B*U,s,0,t);
incre = .01;   p= .01;
% Generating propagated fuzzy state
for i=1 : 100
    level = subs(x ,p);
    xdata([2*i-1 2*i],1) = level;
    ydata([2*i-1 2*i],1) = [p p]';
    p = p+incre;
end
% Plot for propagated membership functions at
% desired time instants.
figure
 plot(xdata,ydata,'.');
 xlim([-2.5 2.5]);
 ylim([0 1])
% Plot of state and control function at "alpha = .5".
t=sym('t')
delta = sqrt(1-(1/log(exp(1)/.5)))
x0 = [-delta/2;delta/2];
x1 = [-2*delta;2*delta];
u = B'*expm(A'*(-t))*inv(W)*(expm(A*(-T))*x1 - x0);
U = subs(u,s);
x =(expm(A*t))*x0 + int(expm(A*(t-s))*B*U,s,0,t);
```

```
tdata = linspace(0,1,200)
odata = zeros(2,200)
udata = zeros(2,200)
subs(x,.5)
for i=1 : size(tdata,2)
    odata(:,i) = subs(x,tdata(i));
    udata(:,i) = subs(u,tdata(i));
end
figure
plot(tdata,odata(1,:),'.',tdata,odata(2,:),'.')
figure
plot(tdata,udata(1,:),'.',tdata,udata(2,:),'.')
```

The following $m-$script is used in demonstrating fuzzy controllability and to compute fuzzy-controllable initial fuzzy states for Example 4.2.21. We have used the results of Theorem 4.2.18 in this script.

```
clear all;
clc;
% Data for target fuzzy states
x1 = -1:.0005:1;
y1 = exp(1).*exp(-(1./(1-(x1.^2))));
x2 = .5:.0005:1.5;
y2 = exp(1).*exp(-((1/4)./((1/4)-((x2-1).^2))));
t = sym('t');
s = sym('s');
T=1;
% System matrices A and B
A1 = [.5 .1;.1 .3];
B1 = [1 -1]';
% Initial and target crisp states
x10 = [2 3]';
x11 = [0 1]';
```

```
W = int((expm(A1*(T-t))*B1*B1'*expm(A1'*(T-t))),t,0,T);
subs(W)
% Computation of controller
u = B1'*expm(A1'*(T-t))*inv(W)*(x11-expm(A1*T)*x10);
U=subs(u,s);
% System matrices after flip operations
A = [.5 .1 0 0; .1 .3 0 0; 0 0 .5 .1; 0 0 .1 .3];
B = [1 -1 1 -1]';
a = sym('a');
magicfactor = norm(exp(A1))*sqrt(2);
x0 = [2-((1-1/(log(exp(1)/a)))^.5)/(magicfactor*2)
      3-((1-1/(log(exp(1)/a)))^.5)/(2*magicfactor)
      2+((1-1/(log(exp(1)/a)))^.5)/(2*magicfactor)
     3+((1-1/(log(exp(1)/a)))^.5)/(2*magicfactor)]';
incre = .01; p= .01;
% state vector computation
x = (expm(A*1))*x0 + int(expm(A*(1-s))*B*U,s,0,1);
% Generating various fuzzy states
for i=1 : 100
   x_1 = x([1 3],:);
   x_2 = x([2 4],:);
   level = subs(x_1,p);
   level_1 =  subs(x_2,p);
   x1_a([2*i-1 2*i],1) = level;
   x2_a([2*i-1 2*i],1) = level_1;
   y1_a([2*i-1 2*i],1) = [p p]';
   y2_a([2*i-1 2*i],1) = [p p]';
% Plot for initial grade functions
   z1_a = x0([1 3],:);
   z2_a = x0([2 4], :);
   i1 = subs(z1_a,p);
   i2 = subs(z2_a,p);
```

```
   t1_a([2*i-1 2*i],1) = i1;
   t2_a([2*i-1 2*i],1) = i2;
   p  = p+incre;
end
% Plot for computed initial state vector x(1)
plot(t1_a,y1_a,'.');
pause
hold on
% Plot for computed initial state vector x(2)
plot(t2_a,y2_a,'.');
hold off
% Propagated state vector x(1) at time t= 1
plot(x1_a,y1_a,'.');
hold on
% Prescribed or desired fuzzy state x(1) at time t=1
plot(x1,y1)
pause
% Propagated state vector x(2) at time t= 1
plot(x2_a,y2_a,'.');
hold on ;
% Prescribed or desired fuzzy state x(2) at time t=1
plot(x2,y2)
```

A.3 M-scripts for Controllability of Nonlinear Matrix Lyapunov Systems

In Example 5.4.13, the following m-script is employed to compute the norms for various operators used in Theorem 5.4.8 of Chapter 5.

```
% Enter the matrix A, B, F,
clc
```

```
clear all
A = [1 2; 3 2];
B = [1 1; 2 1]
F = [1; 1]
alpha = 2/140000
t = sym('t');
s = sym('s');
disp('The matrix A1 is')
A1 = kron(B',eye(2)) + kron(eye(2), A)
B1 = kron(eye(2),F)
% Final time.
T = .1
% Check if the syatem is controllable (rank should be 4))
rank ([B1 A1*B1 A1^2*B1 A1^3*B1])
b = norm(B1);
% The constant k for norm of the operator K
k = int((sum(sum(expm(A1*t))))^2*
(int((sum(sum(expm(-A1*s))))^2,s,0,t)),t,0,T);
normK=sqrt(double(k))
% The constant for the norm of the operator H
normH=b*normK
gramm = int((expm(A1*(T-t))*B1*B1'*expm(A1'*(T-t))),t,0,T);
gr = double(gramm)
eig(gr)
% Norm of the inverse of controllability grammian
c=norm(inv(double(gramm)))
k0 = norm(expm(A1*T));
k1 = double((int((sum(sum(expm(-A1*t))))^2,t,0,T)))
k2 = double(sqrt(k1)*k0)
disp('The norm for R is')
% Norm of the operator $R$.
normR= double(b*k2^2*c)
```

```
% The following value should be less than 1.
double(k*alpha)
% The following critical constant should be
% less then 1.
double(normR*normH*alpha*(1\(1-k*alpha)))
```

LIST OF PAPERS BASED ON THESIS

Papers in Refereed International Journals

1. **Bhaskar Dubey** and Raju K. George (2013). A note on the evolution of solutions of a system of ordinary differential equations with fuzzy initial conditions and fuzzy-inputs. *Journal of Uncertain Systems*, 7(4), 294-302.

2. **Bhaskar Dubey**, Raju K. George (2013). Controllability of semilinear matrix Lyapunov systems. *Electronic Journal of Differential Equations*, 2013(42), 1-12.

3. **Bhaskar Dubey**, Raju K. George (2012). Estimation of controllable initial fuzzy states of linear time-invariant dynamical systems. *Communications in Computer and Information Science*, 283, 316-324.

4. **Bhaskar Dubey**, Raju K. George (2012). On the system of fuzzy differential equations, *Advances in Fuzzy Sets and Systems*, 13(1), 61-75.

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1. **Bhaskar Dubey** and Raju K. George (2013). Controllability of linear time-invariant dynamical systems with fuzzy initial condition, *In Proceedings of the World Congress on Engineering and Computer Science*, San-Francisco, U.S.A., October 23-25, 2013, volume 2, pp. 879-884.

Papers Submitted for Publication

1. B. Dubey, R.K. George, Controllability of impulsive matrix Lyapunov systems, *Communicated to "IEEE Transactions on Automatic Control"*.

2. B. Dubey, R.K. George, A note on the controllability of fuzzy differential dynamical systems, *Communicated to "Inter. J. of Uncertainty Fuzziness and Knowledge Based Systems"*.

3. B. Dubey, R.K. George, A survey on controllability of fuzzy dynamic systems, *Communicated to "IEEE Transactions on Fuzzy Systems"*.

4. B. Dubey, R.K. George, On the solution of linear time-varying differential dynamical systems with fuzzy initial conditions and fuzzy inputs, *Communicated to "Annals of Fuzzy Mathematics and Informatics"*.