The Method of Multiple Scales

A. Salih
Department of Aerospace Engineering
Indian Institute of Space Science and Technology, Thiruvananthapuram
1 September 2014

1 Introduction

Some natural processes have more than one characteristic length or time scales associated with them, for example, the turbulent flow consists of various length scales of the turbulent eddies along with the length scale of the objects over which the fluid flows. The failure to recognize a dependence on more than one space/time scale is a common source of nonuniformity in perturbation expansions.

The *method of multiple scales* (also called the *multiple-scale analysis*) comprises techniques used to construct uniformly valid approximations to the solutions of perturbation problems in which the solutions depend simultaneously on widely different scales. This is done by introducing fast-scale and slow-scale variables for an independent variable, and subsequently treating these variables, fast and slow, as if they are independent.

We will begin by describing the straightforward expansion method and the Poincaré-Lindstedt method for the linear damped oscillator. We then describe the method of multiple scales for the same problem.

2 The Linear Damped Oscillator

We consider the differential equation for the linear damped mass-spring system with no external forces. The equation for displacement $y(\tau)$ is

$$my'' + cy' + ky = 0 \tag{1a}$$

where 'prime' denotes the differentiation with respect to τ. If initially the mass is released from a positive displacement y_i with no initial velocity, we have the following initial conditions:

$$y(0) = y_i, \qquad y'(0) = 0 \tag{1b}$$

We assume here that $c \ll m, k$. Choosing y_i and $\sqrt{m/k}$ as the characteristic distance and characteristic time respectively, we define the following dimensionless variables

$$x = \frac{y}{y_i}, \qquad t = \frac{\tau}{\sqrt{m/k}}$$

Under this change of variables the dimensionless form of the differential equation (1a) and initial conditions (1b) become

$$x'' + 2\varepsilon x' + x = 0 \tag{2a}$$

$$x(0) = 1, \qquad x'(0) = 0 \tag{2b}$$

where

$$\varepsilon = \frac{c}{2\sqrt{mk}} \ll 1$$

is a dimensionless parameter. This equation corresponds to a linear oscillator with weak damping, where the time variable has been scaled by the period of the undamped system. This is the classical example used to illustrate the method of multiple scales.

The exact solution

The exact solution of system (2) is given by

$$x(t) = e^{-\varepsilon t}\left(\cos\sqrt{1-\varepsilon^2}\,t + \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\sin\sqrt{1-\varepsilon^2}\,t\right) \tag{3}$$

note that if the oscillation is undamped, i.e., if $\varepsilon = 0$, we have the following exact solution,

$$x(t) = \cos t$$

where both amplitude and phase of the oscillation remain constant. However, with the presence of damping, (3) shows that both amplitude and phase change with time. In fact, the amplitude drifts on the time scale ε^{-1}, while the phase drifts on the longer time scale ε^{-2}. Note that both the amplitude and phase times scales (ε^{-1} and ε^{-2}) are much longer than the time scale of 1 for the basic oscillation. Of course in this example there is not much amplitude left by the time that the phase has slipped significantly.

By looking at the solution (3), we can say that

$$x = \cos t + O(\varepsilon) \qquad \text{for } t = O(1) \tag{4a}$$

is uniformly true, but is not uniformly valid for $t = O(1/\varepsilon)$. If we are interested in times which are $O(1/\varepsilon)$ then the combination εt must be preserved in the exponential function. Then it is uniformly valid to state that

$$x = e^{-\varepsilon t}\cos t + O(\varepsilon) \qquad \text{for } t = O(1/\varepsilon) \tag{4b}$$

If we are interested in values of t which are $O(1/\varepsilon^2)$ then (4b) is no longer valid. In this case terms of the form $\varepsilon^2 t$ must be preserved in the cosine function appearing in (3). Using binomial expansion, we have

$$\sqrt{1-\varepsilon^2} = 1 - \frac{\varepsilon^2}{2} - \frac{\varepsilon^4}{8} - \frac{\varepsilon^6}{16} - \cdots$$

Thus

$$x = e^{-\varepsilon t}\cos\left(1 - \frac{\varepsilon^2}{2}\right)t + O(\varepsilon) \qquad \text{for } t = O(1/\varepsilon^2) \tag{4c}$$

That is (4c) is uniformly valid for $t = O(1/\varepsilon^2)$.

Notice that if we are concerned only with uniformly valid leading order expansions then the second member of the bracket in (3) never contributes since it is uniformly of $O(\varepsilon)$ for all t.

3 Straightforward Expansion

We will first develop a straight forward expansion for (2) and discuss its nonconformity. So we look for straightforward expansion of an asymptotic solution as $\varepsilon \to 0$,

$$x(t) \sim x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots \tag{5}$$

Substituting (5) into the differential equation (2a) yields

$$x_0'' + \varepsilon x_1'' + \varepsilon^2 x_2'' + \cdots + 2\varepsilon(x_0' + \varepsilon x_1' + \varepsilon^2 x_2' + \cdots) + x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots = 0$$

Collecting coefficients of equal powers of ε gives

$$x_0'' + x_0 + \varepsilon(x_1'' + 2x_0' + x_1) + \varepsilon^2(x_2'' + 2x_1' + x_2) + \cdots = 0$$

Equating coefficients of like powers of ε to 0, gives a sequence of linear differential equations

$$x_0'' + x_0 = 0, \qquad x_0(0) = 1, \quad x_0'(0) = 0 \tag{6a}$$

$$x_1'' + x_1 = -2x_0', \qquad x_1(0) = 0, \quad x_1'(0) = 0 \tag{6b}$$

$$x_2'' + x_2 = -2x_1', \qquad x_2(0) = 0, \quad x_2'(0) = 0 \tag{6c}$$

The respective initial conditions are also shown alongside the equations above. The equation (6a) is the unperturbed problem obtained by setting $\varepsilon = 0$. It is the governing equation of a harmonic oscillator with angular frequency of unity. The solution is

$$x_0 = \cos t$$

Then (6b) becomes

$$x_1'' + x_1 = 2\sin t, \qquad x_1(0) = 0, \quad x_1'(0) = 0 \tag{7}$$

The solution of the nonhomogeneous differential equation (7) is given by

$$x_1 = x_c + x_p$$

where x_c is the general solution of the corresponding homogeneous equation (complementary function of (7))

$$x_1'' + x_1 = 0$$

and x_p is a particular solution of (7). We have

$$x_c = A\cos t + B\sin t$$

The right-hand side of differential equation (7) is of the same form as the general solution of the corresponding homogeneous equation so that a trail particular solution of the form

$$x_p = C_1 t\cos t + C_2 t\sin t$$

must be sought. The constants C_1 and C_2 can be found by the method of undetermined coefficients. Substituting x_p into (7) yields

$$-2C_1\sin t + 2C_2\cos t = 2\sin t$$

Equating like terms gives $C_1 = -1$ and $C_2 = 0$. Thus the general solution of (7) is

$$x_1 = A\cos t + B\sin t - t\cos t$$

Applying the initial conditions on x_1 gives $A = 0$ and $B = 1$. Thus, the solution of (7) is given by

$$x_1 = \sin t - t\cos t$$

Therefore, a two-term approximate solution of (2a) takes the form

$$x(t) = \cos t + \varepsilon(\sin t - t\cos t) \tag{8}$$

The straightforward expansion is not valid when $t > 0(1/\varepsilon)$ due the presence of secular terms. It can be shown that the secular term become more compounded for higher-order expansions. The two-term approximation has a linear secular term, whereas the three-term approximation would have a quadratic secular term.

The two-term expansion (8) can be constructed from the exact solution (3) by expanding the exponential, square root, and trigonometric functions. Nonuniformities are generated in forming the expansions of the exponential term $e^{-\varepsilon t}$ and trigonometric functions $\cos\sqrt{1-\varepsilon^2}\,t$ and $\sin\sqrt{1-\varepsilon^2}\,t$ in powers of ε. We note that the straightforward expansion (8) forces the frequency to be unity, which is independent of the damping. In fact, the presence of the damping changes the frequency from 1 to $\sqrt{1-\varepsilon^2}$. Thus, any expansion procedure that does not account for the dependence of the frequency on ε will fail for large t.

4 Poincaré-Lindstedt Method

We will now apply the Poincaré-Lindstedt Method to the initial value problem (2) to see whether the method is capable of avoiding the secular term that ruined the approximation when a straightforward application of the regular perturbation method is used. The important observation from the earlier analysis is that the breakdown in the straightforward expansion is due to its failure to account for the nonlinear dependence of frequency on the nonlinearity. To account for the fact that the frequency is a function of ε, we let

$$\rho = \omega t$$

where ρ is called the *strained coordinate* and ω is a constant that depends on ε. Then we need to change the independent variable for t to ρ. Using the chain rule, we transform the derivatives according to

$$\begin{aligned}\frac{d}{dt} &= \frac{d}{d\rho}\frac{d\rho}{dt} = \omega\frac{d}{d\rho}\\ \frac{d^2}{dt^2} &= \omega\frac{d^2}{dt\,d\rho} = \omega\frac{d\rho}{dt}\frac{d^2}{d\rho^2} = \omega^2\frac{d^2}{d\rho^2}\end{aligned}$$

Hence, (2) becomes

$$\omega^2 x'' + 2\omega\varepsilon x' + x = 0 \tag{9a}$$

$$x(0) = 1, \qquad x'(0) = 0 \tag{9b}$$

where $x = x(\rho)$ the prime indicate the derivative with respect to ρ. We now try expanding x and ω in powers of ε, i.e.,

$$\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots \tag{10}$$
$$x = x_0(\rho) + \varepsilon x_1(\rho) + \varepsilon^2 x_2(\rho) + \cdots \tag{11}$$

Note that the first term in (10) is unity, which is the unperturbed (undamped) frequency. Substituting (10) into differential equation (9a) to yield

$$(1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots)^2 x'' + 2(1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots)\varepsilon x' + x = 0 \tag{12}$$

Now, substituting the expansion (11) into differential equation (12) and the initial conditions (9b) gives

$$\begin{aligned}(1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots)^2(x_0'' + \varepsilon x_1'' + \varepsilon^2 x_2'' + \cdots) + \\ 2\varepsilon(1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots)(x_0' + \varepsilon x_1' + \varepsilon^2 x_2' + \cdots) + (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) = 0\end{aligned}$$

and

$$x_0(0) + \varepsilon x_1(0) + \varepsilon^2 x_2(0) + \cdots = 1, \qquad x_0'(0) + \varepsilon x_1'(0) + \varepsilon^2 x_2'(0) + \cdots = 0$$

The differential equation above can be written as

$$x_0'' + x_0 + \varepsilon(x_1'' + 2\omega_1 x_0'' + 2x_0' + x_1) + \varepsilon^2(x_2'' + x_2 + 2\omega_2 x_0'' + \omega_1^2 x_0'' + 2\omega_1 x_1'' + 2\omega_1 x_0' + 2x_1') + \cdots = 0 \tag{13}$$

$$x_0'' + x_0 = 0, \qquad x_0(0) = 1, \quad x_0'(0) = 0 \tag{14a}$$
$$x_1'' + x_1 = -2\omega_1 x_0'' - 2x_0', \qquad x_1(0) = 0, \quad x_1'(0) = 0 \tag{14b}$$
$$x_2'' + x_2 = -2\omega_2 x_0'' - \omega_1^2 x_0'' - 2\omega_1 x_1'' - 2\omega_1 x_0' - 2x_1', \qquad x_2(0) = 0, \quad x_2'(0) = 0 \tag{14c}$$

The $O(1)$ system (14a) has the solution

$$x_0(\rho) = \cos\rho \tag{15}$$

Then the $O(\varepsilon)$ equation (14b) becomes

$$x_1'' + x_1 = 2\omega_1\cos\rho + 2\sin\rho, \qquad x_1(0) = 0, \quad x_1'(0) = 0 \tag{16}$$

The solution of nonhomogeneous differential equation (16) is given by

$$x_1 = x_c + x_p$$

where x_c is the general solution of the homogeneous equation given by

$$x_c = A\cos\rho + B\sin\rho$$

and x_p is a particular solution of (16) given by

$$x_p = C_1\rho\cos\rho + C_2\rho\sin\rho$$

The constants C_1 and C_2 can be found by the method of undetermined coefficients. Substituting x_p into (16) and equating like terms gives $C_1 = -1$ and $C_2 = \omega_1$. Thus the general solution of (16) is

$$x_1 = A\cos\rho + B\sin\rho - \rho\cos\rho + \omega_1\rho\sin\rho$$

Applying the initial conditions on x_1 gives $A = 0$ and $B = 1$. Thus, the particular solution of (16) is given by

$$x_1(\rho) = \sin\rho - \rho\cos\rho + \omega_1\rho\sin\rho \tag{17}$$

Note that the above solution of x_1 contains two secular terms, which makes the expansion breakdown at large ρ. It may be noted that the secular term $\omega_1\rho\sin\rho$ can be eliminated by setting $\omega_1 = 0$, however, the secular term $\rho\cos\rho$ cannot be eliminated as it does not contain any adjustable parameter. Therefore, a two-term approximate solution of (2a) takes the form

$$x(\rho) = x_0(\rho) + \varepsilon x_1(\rho) = \cos\rho + \varepsilon(\sin\rho - \rho\cos\rho) \tag{18}$$

If we continue further by setting $\omega_1 = 0$ in (14c), it can be easily shown that the solution of (14c) will provide a condition $\omega_2 = 0$. This shows that our attempt to expand ω in terms of ε has failed and consequently, we get $\rho = t$. Thus, the Poincaré-Lindstedt method has failed to yield a perturbation approximation for the linear damped oscillator (2). The reason for the failure of this technique to yield a uniform solution is our insistence on a uniform solution having a constant amplitude as in (15). Since the amplitude is $e^{-\varepsilon t}$ according to the exact solution (3), the only constant-amplitude uniform solution is the one attained after a long time, i.e., steady state. Therefore, although the Poincaré -Lindstedt technique is effective in determining periodic solutions, they are incapable of determining transient responses.

5 The Method of Multiple Scales

Any asymptotic expansion of (3) must simultaneously depict both the decaying and oscillatory behaviors of the solution in order to be uniformly valid in $t = O(1/\varepsilon^k)$. It is clear that the Poincaré-Lindstedt method fails to achieve this. The Poincaré-Lindstedt method provides a way to construct asymptotic approximations of periodic solutions, but it cannot be used to obtain solutions that evolve aperiodically on a slow time-scale. The method of multiple scales is a more general approach that involve two key tricks. The first is the idea of introducing scaled space and time coordinates to capture the slow modulation of the pattern, and treating these as separate variables in addition to the original variables that must be retained to describe the pattern state itself. This is essentially the idea of multiple scales. The second is the use of what are known as solvability conditions in the formal derivation.

We note from analytical solution (3) that the functional dependence of x on t and ε is not disjoint because x depends on the combination of εt as well as on the individual t and ε. Thus in place of $x = x(t;\varepsilon)$, we write

$$x = \hat{x}(t, \varepsilon t; \varepsilon)$$

We return to the regular expansion (8) and rewrite it as

$$x(t) = \cos t + \varepsilon \sin t - \varepsilon t \cos t \tag{19}$$

As in the case of analytical solution, regular expansion also shows that x depends on the combination of εt as well as on the individual t and ε. The trouble with the naive regular expansion is that the small damping changes both the amplitude of the oscillation on a time scale ε^{-1} and the phase of the oscillation on a time scale ε^{-2} by the slow accumulation of small effects. Thus the oscillator has three processes acting on their on time scales. Fist, there is the basic oscillation on the time scale of 1 from the inertia causing the restoring force to overshoot the equilibrium position. Then there is a small drift in the amplitude on the time scale of ε^{-1} and finally a very small drift in the phase on the time scale of ε^{-2} due to the small friction. We recognize these three time scales by introducing three time variables.

$$\begin{aligned}
T_0 &= t && - \quad \text{the } \textit{fast time} \text{ of the oscillation} \\
T_1 &= \varepsilon t && - \quad \text{the } \textit{slower time} \text{ of the amplitude drift} \\
T_2 &= \varepsilon^2 t && - \quad \textit{even slower time} \text{ of the phase drift}
\end{aligned}$$

The rapidly changing features will then be combined into factors which are functions of T_0, while the slowly changing features will then be combined into factors which are functions of T_1 and T_2. Thus we look for a solution of the form

$$x(t;\varepsilon) = x(T_0, T_1, T_2; \varepsilon)$$

In general, if we choose n time scales for the expansion, we look for a solution of the form

$$x(t;\varepsilon) = x(T_0, T_1, T_2, \cdots T_n; \varepsilon)$$

where the time scales are defined as

$$T_0 = t, \qquad T_1 = \varepsilon t, \qquad\qquad T_2 = \varepsilon^2 t, \quad \cdots, \quad T_n = \varepsilon^n t$$

Thus, instead of determining x as a function of t, we determine x as a function of $T_0, T_1, \cdots, T_n$. Note that as real time t increases the fast time T_0 increases at the same rate, while the slower time T_is increase slowly. Using the chain rule we have

$$\begin{aligned}
\frac{d}{dt} &= \frac{\partial}{\partial T_0}\frac{\partial T_0}{\partial t} + \frac{\partial}{\partial T_1}\frac{\partial T_1}{\partial t} + \frac{\partial}{\partial T_2}\frac{\partial T_2}{\partial t} + \cdots \\
&= \frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1} + \varepsilon^2\frac{\partial}{\partial T_2} + \cdots
\end{aligned} \tag{20a}$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial T_0^2} + 2\varepsilon\frac{\partial^2}{\partial T_0 \partial T_1} + \varepsilon^2\left(\frac{\partial^2}{\partial T_0 \partial T_2} + \frac{\partial^2}{\partial T_1^2}\right) + \cdots \tag{20b}$$

Hence, (2) becomes

$$\frac{\partial^2 x}{\partial T_0^2} + 2\varepsilon\frac{\partial^2 x}{\partial T_0 \partial T_1} + \varepsilon^2\left(\frac{\partial^2 x}{\partial T_0 \partial T_2} + \frac{\partial^2 x}{\partial T_1^2}\right) + 2\varepsilon\left(\frac{\partial x}{\partial T_0} + \varepsilon\frac{\partial x}{\partial T_1} + \varepsilon^2\frac{\partial x}{\partial T_2}\right) + x + \cdots = 0 \tag{21a}$$

$$x = 1, \qquad \frac{\partial x}{\partial T_0} + \varepsilon \frac{\partial x}{\partial T_1} + \varepsilon^2 \frac{\partial x}{\partial T_2} + \cdots = 0 \qquad \text{for} \quad T_0 = T_1 = T_2 \cdots = 0 \tag{21b}$$

We note that when $t = 0$, all T_0, T_1, etc. are zero. The benefits of introducing the multiple time variables are not yet apparent. In fact, it appears that we have made the problem harder since the original ordinary differential equation has been turned into a partial differential equation. This is true, but experience with this method has shown that the disadvantages of including this complication are far outweighed by the advantages.

It should be pointed out that the solution of (21) is not unique and that we need to impose more conditions for uniqueness on the solution. This freedom will enable us to prevent secular terms from appearing in the expansion (at least over the time scales we are using). We now seek an asymptotic approximation for x of the form

$$x(t) \equiv x(T_0, T_1, \cdots, T_n; \varepsilon) \sim x_0(T_0, T_1, \cdots, T_n) + \varepsilon x_1(T_0, T_1, \cdots, T_n) + \varepsilon^2 x_2(T_0, T_1, \cdots, T_n) + \cdots \tag{22}$$

It must be understood that there are actually only two independent variables, t and ε, in (22); T_is are functions of these two, and so is not independent. Nevertheless, the principal steps in finding the coefficients x_n are carried out as though $T_0, T_1, \cdots, T_n$ and ε were independent variables. This is one reason why these steps cannot be justified rigorously in advance, but are merely heuristic. Secondly, it must be remarked that (22) is a generalized asymptotic expansion, since (21) enters both through the gauges (which are just the powers of (21)) and also through the coefficients x_n by way of T_i. Although there is no general theorem allowing the differentiation of a generalized asymptotic expansion term by term, it is nevertheless reasonable to construct the coefficients of (22) on the assumption that such differentiation is possible, and then to justify the resulting series by direct error estimation afterwards.

5.1 The first-order two-scale expansion

Before proceeding further, we will first assume that there are only two time scales (T_0 and T_1) involved in the present problem. The scales are defined as

$$T_0 = t, \qquad T_1 = \varepsilon t$$

Thus, instead of determining x as a function of t, we determine x as a function of T_0, T_1. Note that the time T_0 must increase a great deal before the time T_1 will change appreciably, when ε is small. With this, the differential equation and initial conditions given in (21) become

$$\frac{\partial^2 x}{\partial T_0^2} + 2\varepsilon \frac{\partial^2 x}{\partial T_0 \partial T_1} + \varepsilon^2 \frac{\partial^2 x}{\partial T_1^2} + 2\varepsilon \left(\frac{\partial x}{\partial T_0} + \varepsilon \frac{\partial x}{\partial T_1} \right) + x + \cdots = 0 \tag{23a}$$

$$x = 1, \qquad \frac{\partial x}{\partial T_0} + \varepsilon \frac{\partial x}{\partial T_1} = 0 \qquad \text{for} \quad T_0 = T_1 = 0 \tag{23b}$$

We seek an asymptotic approximation for x of the form

$$x(t) \equiv x(T_0, T_1; \varepsilon) \sim x_0(T_0, T_1) + \varepsilon x_1(T_0, T_1) \tag{24}$$

Substituting this into (23a) yields the following:

$$\frac{\partial^2 x_0}{\partial T_0^2} + \varepsilon \frac{\partial^2 x_1}{\partial T_0^2} + 2\varepsilon \frac{\partial^2 x_0}{\partial T_0 \partial T_1} + 2\varepsilon \frac{\partial x_0}{\partial T_0} + x_0 + \varepsilon x_1 + \cdots = 0$$

Collecting coefficients of equal powers of ε gives

$$\frac{\partial^2 x_0}{\partial T_0^2} + x_0 + \varepsilon\left(\frac{\partial^2 x_1}{\partial T_0^2} + 2\frac{\partial^2 x_0}{\partial T_0 \partial T_1} + 2\frac{\partial x_0}{\partial T_0} + x_1\right) = 0 \tag{25}$$

Equating coefficients of like powers of ε to 0, gives the following sequence of linear partial differential equations:

$$O(1): \qquad \frac{\partial^2 x_0}{\partial T_0^2} + x_0 = 0 \tag{26a}$$

$$O(\varepsilon): \qquad \frac{\partial^2 x_1}{\partial T_0^2} + x_1 = -2\frac{\partial^2 x_0}{\partial T_0 \partial T_1} - 2\frac{\partial x_0}{\partial T_0} \tag{26b}$$

It should be remembered that even the step of equating coefficients of equal powers of ε, used in passing from (25) to (26), is not justified by any theorem about generalized asymptotic expansions (since there is no uniqueness theorem for such expansions). It is instead a heuristic assumption used to arrive at a candidate for an approximate solution, whose validity is to be determined afterwards by error analysis.

The respective initial conditions for (26a) and (26b) are given by

$$x_0 = 1, \qquad \frac{\partial x_0}{\partial T_0} = 0 \qquad \text{for} \quad T_0 = T_1 = 0 \tag{27a}$$

$$x_1 = 0, \qquad \frac{\partial x_1}{\partial T_0} = -\frac{\partial x_0}{\partial T_1} \qquad \text{for} \quad T_0 = T_1 = 0 \tag{27b}$$

Since T_0 and T_1 are being treated (temporarily) as independent, the differential equation (26a) is actually a 'partial' differential equation for a function x_0 of two variables T_0 and T_1. However, since no derivatives with respect to T_1 appear in (26a), it may be regarded instead as an 'ordinary' differential equation for a function of T_0 regarding T_1 as merely an auxiliary parameter. Therefore the general solution of (26a) may be obtained from the general solution of the corresponding ordinary differential equation just by letting the arbitrary constants become arbitrary functions of T_1. Thus the general solution of (26a) can be written as

$$x_0 = A_0(T_1)\cos T_0 + B_0(T_1)\sin T_0 \tag{28}$$

in which A_0 and B_0 are constant as far as the fast T_0 variations are concerned, but are allowed to vary over the slow T_1 time. The initial conditions give

$$A_0(0) = 1 \qquad \text{and} \qquad B_0(0) = 0 \tag{29}$$

We have used all of the information contained in (26a) & (27a), and the functions A_0 and B_0 are still undetermined except for their initial values (29). In order to complete the determination of these functions, and hence of x_0, we must consider the next order of approximation, i.e., $O(\varepsilon)$. This is accomplished by considering the equation (26b). From (28), we have

$$\frac{\partial x_0}{\partial T_0} = -A_0(T_1)\sin T_0 + B_0(T_1)\cos T_0$$

and

$$\frac{\partial^2 x_0}{\partial T_1 \partial T_0} = \frac{\partial}{\partial T_1}\left(\frac{\partial x_0}{\partial T_0}\right) = -\sin T_0 \frac{\partial A_0}{\partial T_1} + \cos T_0 \frac{\partial B_0}{\partial T_1}$$

Substituting the above relations in (26b), we obtain

$$\frac{\partial^2 x_1}{\partial T_0^2} + x_1 = 2\left(\frac{\partial A_0}{\partial T_1} + A_0\right)\sin T_0 - 2\left(\frac{\partial B_0}{\partial T_1} + B_0\right)\cos T_0 \tag{30}$$

Since both the right-hand side of (30) and the complementary function of this equation contain terms proportional to $\sin T_0$ & $\cos T_0$, the particular solution of x_1 will have secular terms in it. Thus, to obtain a uniform expansion each of the coefficients of $\sin T_0$ & $\cos T_0$ must independently vanish. The vanishing of these coefficients yields the condition for the determination of A_0 and B_0. Hence

$$\frac{\partial A_0}{\partial T_1} + A_0 = 0 \tag{31}$$

$$\frac{\partial B_0}{\partial T_1} + B_0 = 0 \tag{32}$$

Equations (31) and (32) represent the conditions to avoid secular terms in x_1. The solution of (31) and (32) are

$$A_0 = a_0 e^{-T_1} \tag{33}$$

$$B_0 = b_0 e^{-T_1} \tag{34}$$

where a_0 and b_0 are constants of integration. To obtain x_0, we substitute (33) and (34) in (28) to obtain

$$x_0 = a_0 e^{-T_1}\cos T_0 + b_0 e^{-T_1}\sin T_0 \tag{35}$$

We can now impose the initial conditions for x_0 given in (27a), which is repeated below:

$$x_0(0,0) = 1, \qquad \frac{\partial x_0}{\partial T_0}(0,0) = 0$$

Imposing these on the general solution (35) yields

$$a_0 = 1 \quad \text{and} \quad b_0 = 0$$

and thus we obtain the solution

$$x_0 = e^{-T_1}\cos T_0$$

Note that we did not evaluate x_1 but merely ensure that secular terms are avoided so that we may write

$$x_0 = e^{-T_1}\cos T_0 + O(\varepsilon) \tag{36}$$

In terms of the original variables, x becomes

$$x = e^{-\varepsilon t}\cos t + O(\varepsilon) \tag{37}$$

which is uniformly valid for $t = O(1/\varepsilon)$ and in agreement with exact solution (3) to $O(\varepsilon)$.

5.2 Higher-order approximations

Different strategies for finding higher order multiple scale approximations are available in the literature. One of the simple strategies is to continue with the two-scale method using $T_0 = t$ and $T_1 = \varepsilon t$. But the solution is written in the form $x(t) \sim x_0(T_0,T_1) + \varepsilon x_1(T_0,T_1) + \varepsilon^2 x_2(T_0,T_1) + \cdots$. The theory of higher order approximations by the first of these methods is fairly well understood. This form continues to use the two time scales and is applicable to a wide variety of problems. The purpose of this strategy is to improve the accuracy of the first approximation to any order of accuracy by computing higher order terms in the series (22), without attempting to increase the length of time $O(1/\varepsilon)$ for which the approximation is valid. In another strategy, multiple scale methods using three scales $T_0 = t$, $T_1 = \varepsilon t$, and $T_2 = \varepsilon^2 t$ are being used. The solutions are written $x(t) \sim x_0(T_0,T_1,T_2) + \varepsilon x_1(T_0,T_1,T_2) + \varepsilon^2 x_2(T_0,T_1,T_2)$. A variation of this method (the "short form") omits one time scale in each successive term; for instance, a three-scale three-term solution would look like $x(t) \sim x_0(T_0,T_1,T_2) + \varepsilon x_1(T_0,T_1) + \varepsilon^2 x_2(T_0)$.

At least in theory, the first strategy is always successful at achieving its goal. However, to carry out the solution in practice requires solving certain differential equations in order to eliminate secular terms; these differential equations are in general nonlinear, and therefore may not have "closed form" solutions (that is, explicit solutions in terms of elementary functions). The second strategy of using three time scales is more general and ambitious but less satisfactory. Their aim is not only to improve the asymptotic order of the error estimate, but also to extend the validity of the approximations to "longer" intervals of time, that is, expanding intervals of length $O(\varepsilon^2)$ or longer. This form tries improves the accuracy to second order and at the same time valid over the length of time $O(1/\varepsilon^2)$. It should be pointed that these methods were originally developed by heuristic reasoning only, and there does not yet exist a fully adequate rigorous theory explaining their range of validity.

5.2.1 The second-order three-time scale expansion

Here we seek an asymptotic approximation for x of the form

$$x(t) \equiv x(T_0,T_1,T_2;\varepsilon) \sim x_0(T_0,T_1,T_2) + \varepsilon x_1(T_0,T_1,T_2) + \varepsilon^2 x_2(T_0,T_1,T_2) \tag{38}$$

for three time scales

$$T_0 = t, \qquad T_1 = \varepsilon t, \qquad T_2 = \varepsilon^2 t$$

Substituting (38) into (21a) yields the following:

$$\frac{\partial^2 x_0}{\partial T_0^2} + \varepsilon\frac{\partial^2 x_1}{\partial T_0^2} + \varepsilon^2\frac{\partial^2 x_2}{\partial T_0^2} + 2\varepsilon\frac{\partial^2 x_0}{\partial T_0\partial T_1} + 2\varepsilon^2\frac{\partial^2 x_1}{\partial T_0\partial T_1} + 2\varepsilon^2\frac{\partial^2 x_0}{\partial T_0\partial T_2} + \varepsilon^2\frac{\partial^2 x_0}{\partial T_1^2} \\ + 2\varepsilon\left(\frac{\partial x_0}{\partial T_0} + \varepsilon\frac{\partial x_0}{\partial T_1} + \varepsilon\frac{\partial x_1}{\partial T_0}\right) + x_0 + \varepsilon x_1 + \varepsilon^2 x_2 = 0$$

Collecting coefficients of equal powers of ε gives

$$\frac{\partial^2 x_0}{\partial T_0^2} + x_0 + \varepsilon\left(\frac{\partial^2 x_1}{\partial T_0^2} + 2\frac{\partial^2 x_0}{\partial T_0\partial T_1} + 2\frac{\partial x_0}{\partial T_0} + x_1\right) \\ + \varepsilon^2\left(\frac{\partial^2 x_2}{\partial T_0^2} + 2\frac{\partial^2 x_1}{\partial T_0\partial T_1} + 2\frac{\partial^2 x_0}{\partial T_0\partial T_2} + \frac{\partial^2 x_0}{\partial T_1^2} + 2\frac{\partial x_0}{\partial T_1} + 2\frac{\partial x_1}{\partial T_0} + x_2\right) = 0 \tag{39}$$

Equating coefficients of like powers of ε to 0, gives the following sequence of linear partial differential equations:

$$O(1): \qquad \frac{\partial^2 x_0}{\partial T_0^2} + x_0 = 0 \tag{40a}$$

$$O(\varepsilon): \qquad \frac{\partial^2 x_1}{\partial T_0^2} + x_1 = -2\frac{\partial^2 x_0}{\partial T_0 \partial T_1} - 2\frac{\partial x_0}{\partial T_0} \tag{40b}$$

$$O(\varepsilon^2): \qquad \frac{\partial^2 x_2}{\partial T_0^2} + x_2 = -2\frac{\partial^2 x_1}{\partial T_0 \partial T_1} - 2\frac{\partial^2 x_0}{\partial T_0 \partial T_2} - \frac{\partial^2 x_0}{\partial T_1^2} - 2\frac{\partial x_0}{\partial T_1} - 2\frac{\partial x_1}{\partial T_0} \tag{40c}$$

The respective initial conditions for (40) are given by

$$x_0 = 1, \qquad \frac{\partial x_0}{\partial T_0} = 0 \qquad \text{for} \quad T_0 = T_1 = T_2 = 0 \tag{41a}$$

$$x_1 = 0, \qquad \frac{\partial x_1}{\partial T_0} = -\frac{\partial x_0}{\partial T_1} \qquad \text{for} \quad T_0 = T_1 = T_2 = 0 \tag{41b}$$

$$x_2 = 0, \qquad \frac{\partial x_2}{\partial T_0} = -\frac{\partial x_1}{\partial T_1} - \frac{\partial x_0}{\partial T_2} \qquad \text{for} \quad T_0 = T_1 = T_2 = 0 \tag{41c}$$

It is clear that to solve (40c), we need the solutions of (40a) and (40b). The general solution of (40a) can be written as

$$x_0 = A_0(T_1, T_2)\cos T_0 + B_0(T_1, T_2)\sin T_0 \tag{42}$$

in which A_0 and B_0 are constant as far as the fast T_0 variations are concerned, but are allowed to vary over the slow times T_1 and T_2. The initial conditions give

$$A_0(0,0) = 1 \qquad \text{and} \qquad B_0(0,0) = 0 \tag{43}$$

Here the functions A_0 and B_0 are still undetermined except for their initial values (43). In order to complete the determination of these functions, and hence of x_0, we must consider the next order of approximation, i.e., $O(\varepsilon)$. This is accomplished by considering the equation (40b). From (42), we have

$$\frac{\partial x_0}{\partial T_0} = -A_0(T_1, T_2)\sin T_0 + B_0(T_1, T_2)\cos T_0$$

and

$$\frac{\partial^2 x_0}{\partial T_1 \partial T_0} = \frac{\partial}{\partial T_1}\left(\frac{\partial x_0}{\partial T_0}\right) = -\frac{\partial A_0}{\partial T_1}\sin T_0 + \frac{\partial B_0}{\partial T_1}\cos T_0$$

Substituting the above relations in (40b), we obtain

$$\frac{\partial^2 x_1}{\partial T_0^2} + x_1 = 2\left(\frac{\partial A_0}{\partial T_1} + A_0\right)\sin T_0 - 2\left(\frac{\partial B_0}{\partial T_1} + B_0\right)\cos T_0 \tag{44}$$

Since both the right-hand side of (44) and the complementary function of this equation contain terms proportional to $\sin T_0$ & $\cos T_0$, the particular solution of x_1 will have secular terms in it. Thus, to obtain a uniform expansion each of the coefficients of $\sin T_0$ & $\cos T_0$ must independently vanish. The vanishing of these coefficients yields the condition for the determination of A and B. Hence

$$\frac{\partial A_0}{\partial T_1} + A_0 = 0 \tag{45}$$

$$\frac{\partial B_0}{\partial T_1} + B_0 = 0 \tag{46}$$

Equations (45) and (46) represent the conditions to avoid secular terms in x_1. The solution of (45) and (46) are

$$A_0 = a_0(T_2)e^{-T_1} \tag{47}$$

$$B_0 = b_0(T_2)e^{-T_1} \tag{48}$$

where a_0 & b_0 are the integration constants and are function of T_2. They are determined by eliminating the terms that produce secular terms in the second order problem for x_2. To obtain x_0, we substitute (47) and (48) in (42) to obtain

$$x_0 = a_0(T_2)e^{-T_1}\cos T_0 + b_0(T_2)e^{-T_1}\sin T_0 \tag{49}$$

so that

$$\frac{\partial x_0}{\partial T_0} = -a_0e^{-T_1}\sin T_0 + b_0e^{-T_1}\cos T_0$$

and

$$\frac{\partial^2 x_0}{\partial T_1 \partial T_0} = \frac{\partial}{\partial T_1}\left(\frac{\partial x_0}{\partial T_0}\right) = a_0e^{-T_1}\sin T_0 - b_0e^{-T_1}\cos T_0$$

Substitution of the above derivatives into (40b) yields the following equation for x_1

$$\frac{\partial^2 x_1}{\partial T_0^2} + x_1 = 0 \tag{50}$$

Since this is a homogeneous equation, the general solution is given by

$$x_1 = A_1(T_1,T_2)\cos T_0 + B_1(T_1,T_2)\sin T_0 \tag{51}$$

Having determined x_0 and x_1, each term in the right-hand side of (40c) can be evaluated as follows:

$$\frac{\partial x_0}{\partial T_1} = -a_0e^{-T_1}\cos T_0 - b_0e^{-T_1}\sin T_0$$

$$\frac{\partial^2 x_0}{\partial T_1^2} = a_0e^{-T_1}\cos T_0 + b_0e^{-T_1}\sin T_0$$

$$\frac{\partial^2 x_0}{\partial T_0 \partial T_2} = \frac{\partial}{\partial T_2}\left(\frac{\partial x_0}{\partial T_0}\right) = -\frac{\partial a_0}{\partial T_2}e^{-T_1}\sin T_0 + \frac{\partial b_0}{\partial T_2}e^{-T_1}\cos T_0$$

$$\frac{\partial x_1}{\partial T_0} = -A_1\sin T_0 + B_1\cos T_0$$

$$\frac{\partial^2 x_1}{\partial T_0 \partial T_1} = \frac{\partial}{\partial T_1}\left(\frac{\partial x_1}{\partial T_0}\right) = -\frac{\partial A_1}{\partial T_1}\sin T_0 + \frac{\partial B_1}{\partial T_1}\cos T_0$$

Substituting the above relations in (40c), we get

$$\begin{aligned}\frac{\partial^2 x_2}{\partial T_0^2} + x_2 = {}& 2\left(\frac{\partial A_1}{\partial T_1}\sin T_0 - \frac{\partial B_1}{\partial T_1}\cos T_0\right) + 2\left(\frac{\partial a_0}{\partial T_2}e^{-T_1}\sin T_0 - \frac{\partial b_0}{\partial T_2}e^{-T_1}\cos T_0\right)\\ &- \left(a_0e^{-T_1}\cos T_0 + b_0e^{-T_1}\sin T_0\right) + 2\left(a_0e^{-T_1}\cos T_0 + b_0e^{-T_1}\sin T_0\right) + 2\left(A_1\sin T_0 - B_1\cos T_0\right)\end{aligned}$$

Rearranging the above equation to obtain

$$\frac{\partial^2 x_2}{\partial T_0^2} + x_2 = 2\left(\frac{\partial A_1}{\partial T_1} + A_1 + \frac{\partial a_0}{\partial T_2}e^{-T_1} + \frac{1}{2}b_0 e^{-T_1}\right)\sin T_0 - 2\left(\frac{\partial B_1}{\partial T_1} + B_1 + \frac{\partial b_0}{\partial T_2}e^{-T_1} - \frac{1}{2}a_0 e^{-T_1}\right)\cos T_0 \tag{52}$$

The terms on the right-hand side of (52) produce secular terms because the particular solution is of the form

$$x_{2p} = -\left(\frac{\partial A_1}{\partial T_1} + A_1 + \frac{\partial a_0}{\partial T_2}e^{-T_1} + \frac{1}{2}b_0 e^{-T_1}\right)T_0 \sin T_0 - \left(\frac{\partial B_1}{\partial T_1} + B_1 + \frac{\partial b_0}{\partial T_2}e^{-T_1} - \frac{1}{2}a_0 e^{-T_1}\right)T_0 \cos T_0 \tag{53}$$

Therefore, in order to eliminate these secular terms, we must have the following conditions

$$\frac{\partial A_1}{\partial T_1} + A_1 = -\left(\frac{\partial a_0}{\partial T_2} + \frac{1}{2}b_0\right)e^{-T_1} \qquad \text{and} \qquad \frac{\partial B_1}{\partial T_1} + B_1 = -\left(\frac{\partial b_0}{\partial T_2} - \frac{1}{2}a_0\right)e^{-T_1} \tag{54}$$

It may be noted that it is not required to solve for x_2 in order to arrive at (54). One needs only to inspect (52) and eliminate terms that produce secular terms. The general solutions of (54) are

$$\begin{aligned} A_1(T_1,T_2) &= a_1(T_2)e^{-T_1} - \left(\frac{\partial a_0}{\partial T_2} + \frac{1}{2}b_0\right)T_1 e^{-T_1} \\ B_1(T_1,T_2) &= b_1(T_2)e^{-T_1} - \left(\frac{\partial b_0}{\partial T_2} - \frac{1}{2}a_0\right)T_1 e^{-T_1} \end{aligned} \tag{55}$$

where a_1 and b_1 are integration constants as far as derivatives with respect to T_1 are concerned. Substituting for A_1 and B_1 into (51), we obtain

$$x_1 = \left[a_1 - \left(\frac{\partial a_0}{\partial T_2} + \frac{1}{2}b_0\right)T_1\right]e^{-T_1}\cos T_0 + \left[b_1 - \left(\frac{\partial b_0}{\partial T_2} - \frac{1}{2}a_0\right)T_1\right]e^{-T_1}\sin T_0 \tag{56}$$

Also, we have the following equations for x_0

$$x_0 = a_0 e^{-T_1}\cos T_0 + b_0 e^{-T_1}\sin T_0 \tag{57}$$

Therefore, as $T_1 \to \infty$, although x_0, and $x_1 \to 0$, εx_1 becomes $O(x_0)$ as t increases to $O(1/\varepsilon^2)$. Thus the expansion $x_0 + \varepsilon x_1$ breaks down for t as large as $O(1/\varepsilon^2)$ unless the coefficients of T_1 in the brackets in (56) vanish; i.e., unless

$$\begin{aligned} \frac{\partial a_0}{\partial T_2} + \frac{1}{2}b_0 &= 0 \\ \frac{\partial b_0}{\partial T_2} - \frac{1}{2}a_0 &= 0 \end{aligned} \tag{58}$$

Equation (58) is a set of coupled PDE for a_0 and b_0. To solve this system let us proceed as follows. Differentiate the first of (58) with respect to T_2 yields

$$\frac{\partial^2 a_0}{\partial T_2^2} + \frac{1}{2}\frac{\partial b_0}{\partial T_2} = 0$$

Now, using the second of (58) this can be written as

$$\frac{\partial^2 a_0}{\partial T_2^2} + \frac{1}{4}a_0 = 0 \tag{59}$$

Equation (59) is a homogeneous second-order PDE with constant coefficients, and its general solution can be written as

$$a_0(T_2) = a_{00}\cos(T_2/2) + b_{00}\sin(T_2/2) \tag{60}$$

where a_{00} and b_{00} are the integration constants. In a similar manner we can obtain

$$b_0(T_2) = c_{00}\cos(T_2/2) + d_{00}\sin(T_2/2) \tag{61}$$

It is easy to see that the simultaneous system of equations (58) can be satisfied only when

$$c_{00} = -b_{00} \qquad \text{and} \qquad d_{00} = a_{00}$$

Therefore, equation (61) becomes

$$b_0(T_2) = -b_{00}\cos(T_2/2) + a_{00}\sin(T_2/2) \tag{62}$$

With these results, the equation for x_0 (57) becomes

$$\begin{aligned} x_0 &= [a_{00}\cos(T_2/2) + b_{00}\sin(T_2/2)]\, e^{-T_1}\cos T_0 + [-b_{00}\cos(T_2/2) + a_{00}\sin(T_2/2)]\, e^{-T_1}\sin T_0 \\ &= a_{00}e^{-T_1}\,[\cos T_0\cos(T_2/2) + \sin T_0\sin(T_2/2)] - b_{00}e^{-T_1}\,[\sin T_0\cos(T_2/2) - \cos T_0\sin(T_2/2)] \end{aligned}$$

Using the following trigonometric identities

$$\begin{aligned} \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ \sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \end{aligned}$$

the equation for x_0 can be written as

$$x_0 = a_{00}e^{-T_1}\cos(T_0 - T_2/2) - b_{00}e^{-T_1}\sin(T_0 - T_2/2) \tag{63}$$

With the aid of equation (58) the expressions for A_1 and B_1 given by equation (55) becomes

$$\begin{aligned} A_1(T_1,T_2) &= a_1(T_2)e^{-T_1} \\ B_1(T_1,T_2) &= b_1(T_2)e^{-T_1} \end{aligned} \tag{64}$$

and the equation for x_1 (56) becomes

$$x_1 = a_1e^{-T_1}\cos T_0 + b_1e^{-T_1}\sin T_0 \tag{65}$$

The function $a_1(T_2)$ and $b_1(T_2)$ can be determined by carrying out the expansion to third order

$$\begin{aligned} a_1(T_2) &= a_{11}\cos(T_2/2) + b_{11}\sin(T_2/2) \\ b_1(T_2) &= -b_{11}\cos(T_2/2) + a_{11}\sin(T_2/2) \end{aligned} \tag{66}$$

where a_{11} and b_{11} are the integration constants. With this, the equation for x_1 becomes

$$x_1 = a_{11}e^{-T_1}\cos(T_0 - T_2/2) - b_{11}e^{-T_1}\sin(T_0 - T_2/2) \tag{67}$$

Hence the asymptotic approximation for $x = x_0 + \varepsilon x_1$ is given by

$$x = e^{-T_1}\left[a_{00}\cos(T_0 - T_2/2) - b_{00}\sin(T_0 - T_2/2) + \varepsilon\left(a_{11}\cos(T_0 - T_2/2) - b_{11}\sin(T_0 - T_2/2)\right)\right] \quad (68)$$

We can now impose the initial conditions to determine the constants in the equations for x_0 and x_1. Applying the conditions (41a) gives

$$a_{00} = 1 \qquad \text{and} \qquad b_{00} = 0$$

Thus (63) becomes

$$x_0 = e^{-T_1}\cos(T_0 - T_2/2) \quad (69)$$

Applying the conditions (41b) gives

$$a_{11} = 0 \qquad \text{and} \qquad b_{11} = -1$$

Thus (67) becomes

$$x_1 = e^{-T_1}\sin(T_0 - T_2/2) \quad (70)$$

Hence the asymptotic approximation for $x = x_0 + \varepsilon x_1$ is given by

$$x = e^{-T_1}\left[\cos(T_0 - T_2/2) + \varepsilon\sin(T_0 - T_2/2)\right] \quad (71)$$

In terms of the original variables, x becomes

$$x = e^{-\varepsilon t}\left[\cos\left(t - \tfrac{1}{2}\varepsilon^2 t\right) + \varepsilon\sin\left(t - \tfrac{1}{2}\varepsilon^2 t\right)\right] \quad (72)$$

which is uniformly valid for $t = O(1/\varepsilon^2)$ and in agreement with exact solution (3) to $O(\varepsilon^2)$. Here we have used three time scales and hence this solution is valid only for $t = O(1/\varepsilon^2)$ and we could improve the accuracy of this approximation to second order by computing up to second order terms in the series (22).

6 Bibliography

1. Bellman, R., *Perturbation Techniques in Mathematics, Engineering & Physics*, Dover Publications (2003).
2. Bender, C. M. and Orszag, S. A., *Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory*, Springer (1999).
3. Bush, A. W., *Perturbation Methods for Engineers and Scientists*, CRC Press (1992).
4. de Bruijn, N. G., *Asymptotic Methods in Analysis*, 3^{rd} ed., Dover Publications (1981).
5. Hinch, E. J., *Perturbation Methods*, Cambridge Univ. Press (1995).
6. Holmes, M. H., *Introduction to Perturbation Methods*, Springer-Verlag (1995).
7. Kevorkian, J. and Cole, J. D., *Perturbation Methods in Applied Mathematics*, Springer-Verlag (1981).

8. Kevorkian, J. and Cole, J. D., *Multiple Scale and Singular Perturbation Methods*, Springer (1996).

9. Murdock, J. A., *Perturbation Methods: Theory and Methods*, Wiley-Interscience (1991).

10. Nayfe, A. H., *Introduction to Perturbation Techniques*, Wiley-VCH (1993).

11. Nayfe, A. H., *Perturbation Methods*, John Wiley (2000).

12. Simmonds, J. G. and Mann, J. E., *A First Look at Perturbation Theory*, 2^{nd} ed., Dover Publications (1998).

13. Wasow, W. R., *Asymptotic Expansions for Ordinary Differential Equations*, Dover Publications (1987).