Calculus of Variations

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Lecture-1

In Calculus of Variations, we will study maximum and minimum of a certain class of functions. We first recall some maxima/minima results from the classical calculus.

Maxima and Minima

Let X and Y be two arbitrary sets and $f : X \to Y$ be a well-defined function having domain X and range Y. The function values $f(x)$ become comparable if Y is $\mathbb{R}$ or a subset of $\mathbb{R}$. Thus, optimization problem is valid for real valued functions. Let $f : X \to \mathbb{R}$ be a real valued function having X as its domain. Now $x_0 \in X$ is said to be maximum point for the function f if $f(x_0) \geq f(x) \;\; \forall\, x \in X$. The value $f(x_0)$ is called the maximum value of f. Similarly, $x_0 \in X$ is said to be a minimum point for the function f if $f(x_0) \leq f(x) \;\; \forall\, x \in X$ and in this case $f(x_0)$ is the minimum value of f.

Sufficient condition for having maximum and minimum:

Theorem (Weierstrass Theorem)

Let $S \subseteq \mathbb{R}$ and $f : S \to \mathbb{R}$ be a well defined function. Then f will have a maximum/minimum under the following sufficient conditions.

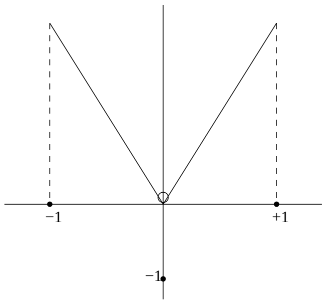
1. $f : S \to \mathbb{R}$ is a continuous function.
2. $S \subset \mathbb{R}$ is a bound and closed (compact) subset of $\mathbb{R}$.

Note that the above conditions are just sufficient conditions but not necessary.

Example 1:

Let $f : [-1, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} -1 & x = 0 \\ |x| & x \neq 0 \end{cases}$$

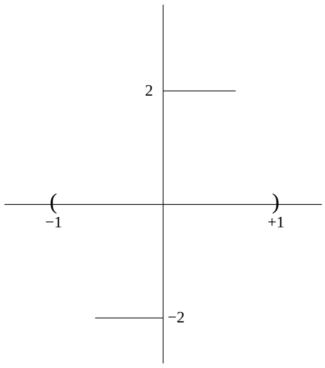


Obviously $f(x)$ is not continuous at $x = 0$. However the $f(x)$ has a minimum point $x_0 = 0$ and maximum points at $x = -1,\ x = 1$. Continuity condition of the Weierstrass theorem is violated but still the function has maximum and minimum.

Example 2:

Consider a function $f : (-1, 1) \to \mathbb{R}$ defined by

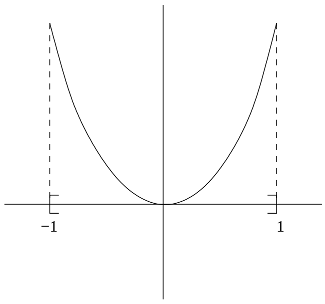
$$f(x) = \begin{cases} -2 & x < 0 \\ 2 & x \geq 0 \end{cases}$$



$f(x)$ has a maximum value $x = 2$ and a minimum value $x = -2$ even though both the conditions (a) and (b) of Weierstrass theorem are violated.

Example 3:

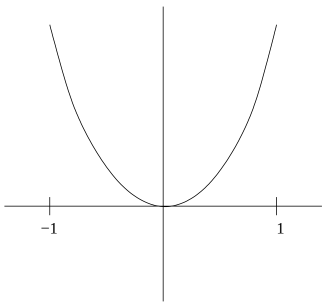
Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) = x^2$.



This function satisfies both the conditions of Weierstrass theorem. $f(x)$ has a minimum value 0 at $x = 0$ and maximum value 1 at $x = -1$ and $x = 1$.

Example 4:

Let $f : (-1, 1) \to \mathbb{R}$ defined by $f(x) = x^2$, f has a minimum at $x = 0$.



But f has no maximum point as $x = -1$ and $x = 1$ are outside the domain of the function. Here the condition (b) is violated.

Necessary condition for Maximum/Minimum when f is differentiable.

Theorem

Let $f : S \to \mathbb{R}$ be a differentiable function and let x_0 be an interior point of S and let x_0 is either a maximum point or minimum point of f. Then the first derivative of f vanishes at x_0.

$$\text{ie} \quad f'(x_0) = 0.$$

This condition is just a necessary condition but not sufficient condition.

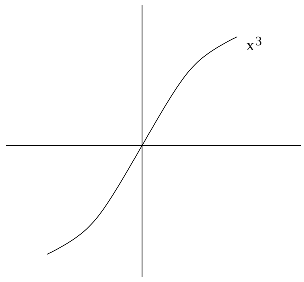
An interior point $x_0 \in D \subseteq \mathbb{R}$ is said to be a stationary point if $f'(x_0) = 0$. A stationary point x_0 need not be maximum point/minimum point. However, if $f''(x_0) > 0$ then x_0 is a minimum point and if $f''(x_0) < 0$ then x_0 is a maximum point.

Example 5:

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^3$$

$$f'(x) = 3x^2 = 0 \quad \text{when} \quad x = 0$$



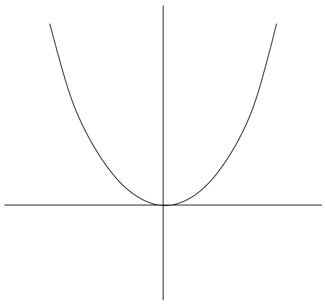
However, $x = 0$ is neither a maximum point nor a minimum point of $f(x) = x^3$.

Example 6:

Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = x^2$$

$$\text{Hence,} \quad f'(x_0) = 2x_0 = 0 \quad \text{when} \quad x_0 = 0.$$



Obviously $x_0 = 0$ is a stationary point and this stationary point is minimum point of $f(x)$, as $f''(x_0) = 2 > 0$.

Necessary conditions for Maxima/Minima functions of several variables

Multi-variable functions.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real valued function of n - variables defined on $\mathbb{R}^n$. If f has partial derivatives at $x_0 \in \mathbb{R}^n$. If x_0 is a maximum point/minimum point of the function $f(x)$ then

$$\left.\frac{\partial f}{\partial x_1}\right|_{x=x_0} = 0, \quad \left.\frac{\partial f}{\partial y_2}\right|_{x=x_0} = 0, \cdots \left.\frac{\partial f}{\partial x_{n-1}}\right|_{x=x_0} = 0$$

A stationary point x_0 is maximum point if the matrix

$$M = \left.\begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}\right|_{x=x_0}$$

is negative definite and x_0 is minimum point if M is positive definite.

Functionals:

Let S be a set of functions. Let $f : S \to \mathbb{R}$ be a real valued function. Such functions are known as a functionals. In otherwords, a functional is a real valued function whose domain is a set of functions.

Example 7:

Let $C[0,1]$ be the set of all continuous functions defined on $[0,1]$
Let $I : C[0,1] \to \mathbb{R}$ be a function defined by

$$I(y) = \int_0^1 y(x)\,dx$$

Obviously $I(y)$ is a functional on $C[0,1]$. The following table gives the values of $I(y)$ for different functions $y(x)$, listed in the table.

$y(x)$	$I(y)$
x	0.5
x^2	0.333
x^3	0.25
$\sin x$	0.4597
$\cos x$	0.8415
e^x	1.7183
1	1

We can find for which function y, the functional $I(y)$ has a maximum value or minimum value. In the above example, $I(y)$ will have minimum value for $y(x) = x^3$ and $I(y)$ will have maximum value for the function $y(x) = e^x$ out of the seven functions given here.

Let $C^1[x_1, x_2]$ denote the set of continuously differentiable functions defined on $[x_1, x_2]$. Now consider a functional $I : C^1[x_1, x_2] \to \mathbb{R}$ defined by $I(y) = \int_{x_1}^{x_2} F(x, y, y')dx$ subject to the end conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.

In calculus of variations the basic problem is to find a function y for which the functional $I(y)$ is maximum or minimum. We call such functions as *extremizing functions* and the value of the functional at the extremizing function as *extremum.*

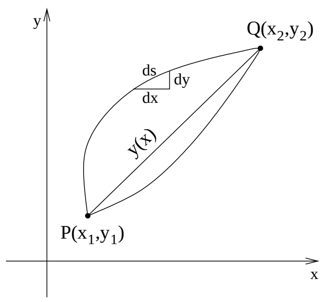
Consider the extremization problem

$$\underset{y}{\text{Extremize}} \quad I(y) = \int_{x_1}^{x_2} F(x, y, y')dx$$

subject to the end conditions $y(x_1) = y_1$ and $y(x_2) = y_2$, where F is a twice continuously differentiable function.

Example

Find the shortest smooth plane curve joining two distinct points $P(x_1, y_1)$ and $Q(x_2, y_2)$.



There are infinitely many functions y passing through the given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$. We are looking for a function which will have minimum arc length. Let ds be a small strip on the curve then we have.

$$\begin{aligned} ds^2 &= dx^2 + dy^2 \\ \left(\frac{ds}{dx}\right)^2 &= 1 + \left(\frac{dy}{dx}\right)^2 \\ ds &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \end{aligned}$$

$$\text{Total arc length } I(y) = \int_P^Q ds = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Thus, the problem is to minimize $I(y)$ subject to the end conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.

Lecture-2

Lemma (Fundamental Lemma of Calculus of Variations)

If $f(x)$ is a continuous function defined on $[a, b]$ and if $\displaystyle\int_a^b f(x)g(x)\ dx = 0$ for every function $g(x) \in C(a, b)$ such that $g(a) = g(b) = 0$ then $f(x) \equiv 0$ for all $x \in [a, b]$.

Proof:

Let $f(x) \neq 0$ for some $c \in (a,b)$. Without loss of generality let us assume that $f(c) > 0$. Now because of continuity of f we have $f(x) > 0$ for some interval $[x_1, x_2] \subset [a,b]$ that contains the point c.

$$\text{Let } g(x) = \begin{cases} (x - x_1)(x_2 - x) & \text{for } \; x \in [x_1, x_2] \\ 0 & \text{outside } \; [x_1, x_2] \end{cases}$$

Note that $(x - x_1)(x_2 - x)$ is positive for $x \in (x_1, x_2)$.
Now consider

$$\begin{aligned} \int_a^b f(x)g(x)\ dx &= \int_a^{x_1} \cancelto{0}{f(x)g(x)\ dx} + \int_{x_1}^{x_2} f(x)g(x)\ dx + \int_{x_2}^{b} \cancelto{0}{f(x)g(x)\ dx} \\ &= \int_{x_1}^{x_2} f(x)g(x)\ dx \\ &= \int_{x_1}^{x_2} f(x)(x - x_1)(x_2 - x)\ dx > 0 \end{aligned}$$

Thus we get a contradiction to what is given in the Lemma. This implies that $f(x) \equiv 0$ on $[a,b]$.

Euler-Lagrange Equation (Necessary Condition for Extremum)

Theorem: If $y(x)$ is an extremizing function for the problem

$$\text{Minimize/Maximize} \quad I(y) \;=\; \int_{x_1}^{x_2} F(x, y, y')\, dx \tag{1}$$

with end conditions $y(x_1) = y_1$ and $y(x_2) = y_2$ then $y(x)$ satisfies the BVP.

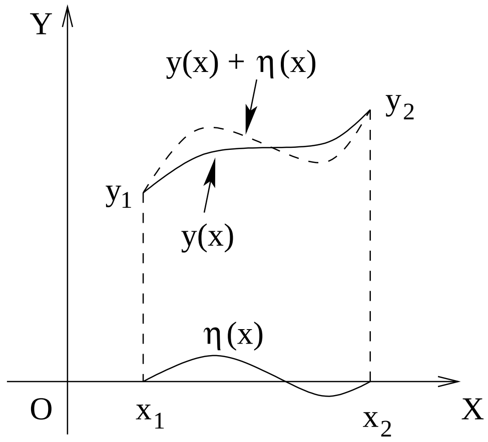
$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} \;=\; 0 \tag{2}$$

$$y(x_1) = y_1 \quad \text{and} \quad y(x_2) = y_2. \tag{3}$$

Equation (2) is known as the **Euler-Lagrange equation**.

Proof:

Let $y(x)$ be an extremizing function for the functional $I(y)$ in (1).



Let $Y = y(x) + \epsilon\ \eta(x)$ be a variation of $y(x)$, where $\eta(x)$ is a continuously differentiable function with $\eta(x_1) = 0 = \eta(x_2)$ and ϵ is a small constant.
Hence I along the path $Y = y(x) + \epsilon\ \eta(x)$ is given by

$$I(\epsilon) = \int_{x_1}^{x_2} F(x,\, y(x) + \epsilon\ \eta(x),\, y'(x) + \epsilon\ \eta'(x))\ dx = \int_{x_1}^{x_2} F(x, Y(x), Y'(x))\ dx$$

Since $y(x)$ is an extremizing function, $I(\epsilon)$ has extremum when $\epsilon = 0$.
Thus, by classical calculus,

$$\begin{aligned} \frac{dI}{d\epsilon}|_{\epsilon=0} &= 0 \\ \Longrightarrow \quad \frac{dI}{d\epsilon} &= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial x}\frac{\partial x}{\partial \epsilon} + \frac{\partial F}{\partial Y}\frac{\partial Y}{\partial \epsilon} + \frac{\partial F}{\partial Y'}\frac{\partial Y'}{\partial \epsilon}\right] dx \end{aligned}$$

But $\dfrac{\partial x}{\partial \epsilon} = 0$ as x is independent of ϵ.

$$\begin{aligned} \frac{\partial Y}{\partial \epsilon} &= \eta(x) \\ \frac{\partial Y'}{\partial \epsilon} &= \eta'(x) \\ \therefore \quad \frac{dI}{d\epsilon} &= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial Y}\eta(x) + \frac{\partial F}{\partial Y'}\eta'(x)\right] dx \end{aligned}$$

Integration by parts we get

$$\begin{aligned} \int_{x_1}^{x_2} \frac{\partial F}{\partial Y'}\eta'(x)\, dx &= \left.\frac{\partial F}{\partial Y'}\eta(x)\right]_{x_1}^{x_2} \to 0 - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial Y'}\right)\eta(x)\, dx \\ &\quad \text{(using } \eta(x_1) = \eta(x_2) = 0) \\ &= -\int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial Y'}\right)\eta(x)\, dx \\ \left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} &= \int_{x_1}^{x_2} \left.\left[\frac{\partial F}{\partial Y} - \frac{d}{dx}\left(\frac{\partial F}{\partial Y'}\right)\right]\right|_{\epsilon=0} \eta(x)\, dx = 0 \\ &= \int_{x_1}^{x_2} \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta(x)\ dx = 0 \\ &\quad \text{as } Y(x)|_{\epsilon=0} = y(x) \text{ and } Y'(x)|_{\epsilon=0} = y'(x) \end{aligned}$$

Since $\eta(x)$ is arbitrary function, by applying the Fundamental Lemma of calculus of variations, we get

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

Different Forms of Euler Lagrange Equation

Suppose that $y(x)$ is an extremizer of $I(y)$.

$$\begin{aligned}
\text{Since } F &= F(x, y, y') \\
\frac{d}{dx}(F) &= \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial y'}\frac{dy'}{dx} \\
&= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y'' \qquad (4) \\
\text{Consider } \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) &= y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y'}y'' \qquad (5)
\end{aligned}$$

Subtracting (4) - (5) we get,

$$\begin{aligned}
\frac{dF}{dx} - \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)y' \\
\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial x} &= y'\cancelto{0}{\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]}
\end{aligned}$$

As $y(x)$ is an extremizer and by using Euler-Lagrange equation we get

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial x} = 0$$

which is another form of Euler-Lagrange Equation.

Special Cases

$$\text{Extremize } I(y) = \int_{x_1}^{x_2} F(x, y, y')\,dx \quad y(x_1) = y_1\,;\, y(x_2) = y_2.$$

(i) When x does not appear in F explicitly

$$\frac{\partial F}{\partial x} = 0$$

Hence $\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial x} = 0$ becomes $\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0$ or $F - y'\frac{\partial F}{\partial y'} = \text{const.}$ This is known as **Beltrami Identity**.

(ii) When y does not appear in F explicitly

$$\frac{\partial F}{\partial y} = 0$$

Hence, $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$ reduces to $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ or $\frac{\partial F}{\partial y'} = \text{const.}$

(iii) When y' does not appear in F explicitly

$$\frac{\partial F}{\partial y'} = 0$$

Then $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$ reduces to $\frac{\partial F}{\partial y} = 0.$

Example-1

Find the shortest smooth plane curve joining two distinct points (x_1, y_1) and (x_2, y_2). We are minimizing the arc length of the function y

$$\begin{aligned}
\text{Minimize } I(y) &= \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
\text{Here, } F(x, y, y') &= \sqrt{1 + y'^2} \\
\frac{\partial F}{\partial y} &= 0\ ; \quad \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}} \\
\text{Euler-Lagrange equation} & \\
\Longrightarrow \quad \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) &= 0 \quad \frac{y'}{\sqrt{1 + y'^2}} = \text{const} = \text{c} \\
y' &= c\sqrt{1 + y'^2} \\
y'^2 &= c^2(1 + y'^2) \\
y'^2 - c^2 y'^2 &= c^2 \\
(1 - c^2) y'^2 &= c^2 \\
\Longrightarrow \quad y'^2 &= \frac{c^2}{1 - c^2} \\
\Longrightarrow \quad y' &= \sqrt{\frac{c^2}{1 - c^2}} = m \\
y &= mx + b \\
y(x_1) = y_1 &\Longrightarrow y_1 = mx_1 + b \\
y(x_2) = y_2 &\Longrightarrow y_2 = mx_2 + b \\
\frac{y_1 - y_2}{x_1 - x_2} = m \quad &; \quad b = y_1 - \left(\frac{y_1 - y_2}{x_1 - x_2}\right) x_1
\end{aligned}$$

Thus the curve having minimum arc length passing through the given two fixed point is a straight line.

Exercise

Show that another form of Euler - Lagrange equation is $F_y - F_{y'x} - F_{y'y}y' - F_{y'y'}y'' = 0$.

Example-2

Find the extremals of the functional $\displaystyle\int_{x_0}^{x_1} \frac{y'^2}{x^3}\, dx$.

$$
\begin{aligned}
F(x,y,y') &= \frac{y'^2}{x^3}\ ; \quad \frac{\partial F}{\partial y} = 0; \quad \frac{\partial F}{\partial y'} = \frac{2y'}{x^3} \\
\text{Euler- Lagrange equation} & \\
\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) &= \frac{d}{dx}\left(\frac{2y'}{x^3}\right) = \frac{2x^3y'' - 6y'x^2}{x^6} = \frac{2}{x^4}\left(xy'' - 3y'\right) = 0 \\
\text{Thus we have} \quad xy'' - 3y' &= 0 \quad \text{or} \quad \frac{y''}{y'} = \frac{3}{x} \\
\text{Integrating} \quad \int \frac{y''}{y'} &= 3\int \frac{1}{x}\, dx + c \\
\ln\, y' &= 3\, \ln\, x + c \\
\ln\left(\frac{y'}{x^3}\right) &= c \quad \text{or} \quad y' = C\ x^3 \\
y &= \frac{Cx^4}{4} + C_2 \\
y &= Ax^4 + B
\end{aligned}
$$

Example-3

$$
\begin{aligned}
\text{Extremize } I(y) &= \int_{x_1}^{x_2} 1 + y'^2\, dx, \quad y(x_1) = y(x_2) = 0 \\
F &= 1 + y'^2, \quad \frac{\partial F}{\partial y'} = 2y' \\
\text{Euler Equation} \quad \frac{d}{dx}\left(2y'\right) &= 0 \quad \Longrightarrow \quad 2y'' = 0 \\
y'(x) &= C\ , \quad y(x) = Cx + D \\
C &= \frac{y_1 - y_2}{x_1 - x_2};
\end{aligned}
$$

Example-4

Find the curve y on which the functional $\displaystyle\int_0^1 y'^2 + 12xy\ dx, y(0) = 0, y(1) = 1$ is exrtremum.

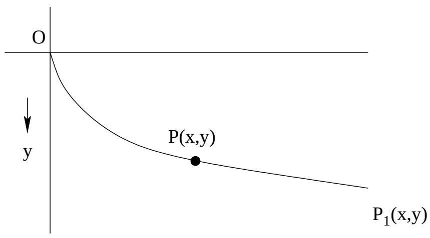
Solution:

$$\begin{aligned} \text{Here } F &= y'^2 + 12xy \\ \text{Euler Lagrange Equation:} \quad 12x - 2y'' &= 0 \quad \text{or} \quad y'' = 6x \\ y' &= 3x^2 + C \\ y &= x^3 + cx + c' \\ \text{Applying the conditions we get} & \\ y &= x^3 \end{aligned}$$

Lecture-3

Brachistochrone Problem (Shortest Time of Descent Problem)

Find the shortest path on which a particle in the absence of friction will slide from one point to another point in the shortest time under the action of gravity.



Solution:

Let the particle slide from o along the path OP_1.
Let at time t, the particle be at $P(x, y)$. Let arc $OP = s$.
By the principle of work and energy, we have
KE at $P - KE$ at $O =$ Work done in moving the particle from O to P.

$$\begin{aligned} \frac{1}{2}m(\frac{ds}{dt})^2 - 0 &= mgy \\ \text{That is} & \\ \frac{ds}{dt} &= \sqrt{2gy} \end{aligned}$$

$\Rightarrow$ Time taken by the particle to move from O to P_1 is given by

$$\begin{aligned} T &= \int_0^T dt = \int_0^{x_1} \frac{ds}{\sqrt{2gy}} = \frac{1}{2g}\int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{y}}dx \\ T &= \frac{1}{\sqrt{2g}}\int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{y}}dx \end{aligned}$$

Brachistochrone Problem is to find y which minimizes the functional

$$I(y) = \frac{1}{\sqrt{2g}} \int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{y}} dx \text{ and } F = \frac{\sqrt{1+y'^2}}{\sqrt{y}}$$

Since x does not appear in F explicitly take the Beltrami Identity.

$$\begin{aligned}
F - y'\frac{\partial F}{\partial y'} &= const. \\
\frac{\sqrt{1+y'^2}}{\sqrt{y}} - y'\frac{\partial}{\partial y'}(\frac{\sqrt{1+y'^2}}{\sqrt{y}}) &= c \\
\frac{\sqrt{1+y'^2}}{\sqrt{y}} - y'\frac{1}{\sqrt{y}}\frac{y'}{\sqrt{1+y'^2}} &= c \\
\frac{1+y'^2-y'^2}{\sqrt{y}\sqrt{1+y'^2}} &= c \\
\frac{1}{\sqrt{y}\sqrt{1+y'^2}} &= c \Rightarrow \sqrt{y(1+y'^2)} = \frac{1}{c} = \sqrt{a}(say) \\
y(1+y'^2) &= a \\
1+y'^2 &= \frac{a}{y} \\
y'^2 &= \frac{a-y}{y} \\
y' &= \sqrt{\frac{a-y}{y}} \\
\frac{dy}{dx} &= \sqrt{\frac{a-y}{y}} \\
\sqrt{\frac{y}{a-y}}dy &= dx \\
\int_0^x dx &= \int_0^y \sqrt{\frac{y}{a-y}}dy
\end{aligned}$$

Since (0,0) is point on the curve, we get c=0.
Let $y= a\sin^2\theta$; $dy = 2a\sin\theta\cos\theta d\theta$

$$\begin{aligned}
\text{Thus, } x &= \int_0^\theta \sqrt{\frac{a\sin^2\theta}{a-a\sin^2\theta}} 2a\sin\theta\cos\theta d\theta \\
x &= \int_0^\theta \frac{\sin\theta}{\cos\theta} 2a\sin\theta\cos\theta d\theta = a\int_0^\theta 2\sin^2 d\theta = a\int_0^\theta (1-\cos 2\theta)d\theta \\
x &= \frac{a}{2}[2\theta - \sin 2\theta]
\end{aligned}$$

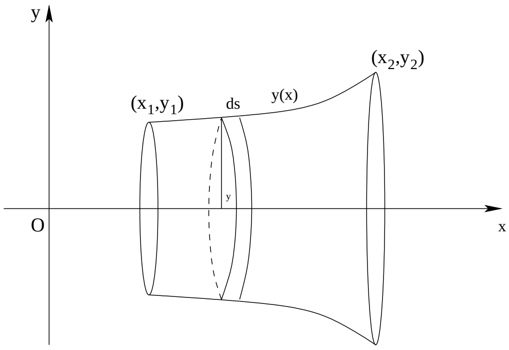
Let $\frac{a}{2} = b$, $2\theta = \phi$

$$x = b(\phi - \sin\phi); y = b(1-\cos\phi)$$

which is a cycloid.

Example - Minimum Surface Area of Rotation.

Find the curve passing through the points (x_1, y_1) and (x_2, y_2) which when rotated about the x-axis gives a minimum surface.



Let ds be a small strip on the curve y. Area of the surface generated by ds when revolved is $2\pi y\ ds$. In the figure the total surface area $= \int_{x_1}^{x_2} 2\pi y ds$

$$= 2\pi \int_{x_1}^{x_2} y\sqrt{(1+y'^2)}dx.$$

This has to be minimum.
Since $F = y\sqrt{(1+y'^2)}$ does not contain x explicitly, thus the Euler's equation reduces to

$$\begin{aligned}
F - y'\frac{\partial F}{\partial y'} &= c : (say) \\
y\sqrt{(1+y'^2)} - y'\frac{\partial}{\partial y'} y\sqrt{(1+y'^2)} &= c \\
\text{i.e} \qquad y\sqrt{(1+y'^2)} - y'\frac{y}{2}(1+y'^2)^{-1/2}.\ 2y' &= c \\
\text{or} \qquad \frac{y}{\sqrt{(1+y'^2)}} &= c \\
y^2 &= c^2 + c^2 y'^2 \\
y' &= \frac{dy}{dx} = \frac{\sqrt{(y^2-c^2)}}{c}
\end{aligned}$$

Separating the variables and integrating, we have

$$\begin{aligned}
\int \frac{dy}{\sqrt{(y^2-c^2)}} &= \int \frac{dx}{c} + c' \\
\cos h^{-1}\frac{y}{c} &= \frac{x+a}{c} \\
\text{i.e} \qquad y &= c\cosh(\frac{x+a}{c})
\end{aligned}$$

which is **catenary**. The constants a and c are determined from the points (x_1, y_1) and (x_2, y_2).

Lecture-4

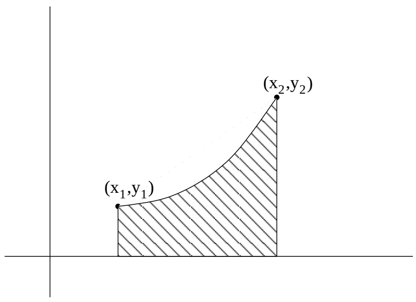
Constrained Extremization Problem

Isoperimetric Problems

In certain problems of calculus of variations, while extremizing a given functional $I(y)$, along with the end conditions $y(x_1) = y_1,\ y(x_2) = y_2$, we also need the extremizing function has to satisfy an additional integral constraint as we see in the following Dido's Problem.

Dido's Problem

Find the plane curve of fixed perimeter which has maximum area above x - axis.



The perimeter and the area under the curve are given by

Perimeter $=$ Arc length $= \displaystyle\int_{x_1}^{x_2} \sqrt{1+y'^2}\ dx$

Area under the curve $A = \displaystyle\int_{x_1}^{x_2} y(x)\ dx.$

Variational Problem:

Maximize $I(y) = \displaystyle\int_{x_1}^{x_2} y(x)\ dx$

subject to the constraints

$\displaystyle\int_{x_1}^{x_2} \sqrt{1+y'^2}\ dx = L$ (given) and with end conditions

$y(x_1) = y_1$ and $y(x_2) = y_2$.

General Problem:

Extremize $I(y) = \displaystyle\int_{x_1}^{x_2} F(x, y, y')\ dx$

subject to the integral constraint

$$\int_{x_1}^{x_2} G(x, y, y')\ dx = L = \text{constant}$$

End conditions are $y(x_1) = y_1$ and $y(x_2) = y_2$.

Lagrange Multiplier Technique:

Convert the constrained optimization problem into an unconstrained optimization problem by the Lagrange Multiplier Technique.

Define a new functional H by $H(x, y, y') = F(x, y, y') + \lambda\, G(x, y, y')$ and optimize $\int_{x_1}^{x_2} H(x, y, y')\ dx$ without constraints.

$$\text{That is,} \quad \text{Optimize } I(y) = \int_{x_1}^{x_2} F(x, y, y') + \lambda\ G(x, y, y')\ dx$$

with end condition $y(x_1) = y_1$ and $y(x_2) = y_2$.

The problem is solved by solving the

Euler Lagrange Equation:

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$$
$$y(x_1) = y_1, \quad y(x_2) = y_2$$

Solution of Dido's Problem:

$$\begin{aligned}
\text{Maximize} \quad I(y) &= \int_{x_1}^{x_2} y(x)\ dx \quad \text{subject to} \\
\int_{x_1}^{x_2} \sqrt{1 + y'^2}\ dx &= L \quad \text{and with end conditions} \\
y(x_1) &= y_1, \quad y(x_2) = y_2 \\
\text{Here,} \quad F(x, y, y') &= y(x), \quad G(x, y, y') = \sqrt{1 + y'^2} \\
H(x, y, y') &= y + \lambda\ \sqrt{1 + y'^2} \\
\frac{\partial H}{\partial y} &= 1, \quad \frac{\partial H}{\partial y'} = \frac{\lambda y'}{\sqrt{1 + y'^2}}
\end{aligned}$$

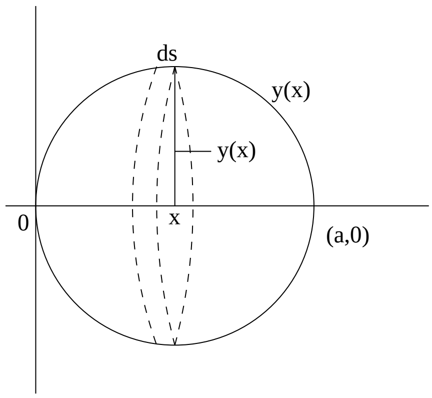
Euler's Equation:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) &= 1 \\
\implies \frac{\lambda y'}{\sqrt{1+y'^2}} &= x+a \\
\implies \frac{\lambda y'}{x+a} &= \sqrt{1+y'^2} \\
\implies \lambda^2 y'^2 &= (1+y'^2)(x+a)^2 \\
\implies y'^2(\lambda^2-(x+a)^2) &= (x+a)^2 \\
y' &= \frac{x+a}{\sqrt{\lambda^2-(x+a)^2}} \\
\implies y &= -\sqrt{\lambda^2-(x+a)^2}+b \\
(y-b)^2 &= \lambda^2-(x+a)^2 \\
\implies (x+a)^2+(y-b)^2 &= \lambda^2
\end{aligned}$$

which is a circle, where the constants a, b, λ can be obtained from 3 conditions namely, two end conditions and one constraint condition.

Problem 2:

Show that sphere is the solid figure of revolution which for a given surface area having maximum volume enclosed.



Consider a small circular strip having height ds and radius y. The surface area is $2\pi y\, ds$.

$$\begin{aligned}
\text{Thus the total surface area of revolution is } S &= \int_0^a 2\pi y \; ds \\
&= \int_0^a 2\pi y\sqrt{1+y'^2} \; dx \\
\text{Volume} &= \int_0^a \pi y^2 \; dx.
\end{aligned}$$

Variational Problem

Maximize $I(y) = \int_0^a \pi y^2 \; dx$
subject to the constraint

$$\int_0^a 2\pi y\sqrt{1+y'^2}\; dx = S \quad \text{(constant)}$$

Define a function $H = F + \lambda\, G = \pi y^2 + \lambda\; 2\pi y\sqrt{1+y'^2}$.
As x is not appearing in H explicitly, we have Euler's Equation (Beltrami Identity)

$$\begin{aligned} H - y'\,\frac{\partial H}{\partial y'} &= \text{const.} \\ \pi y^2 + 2\pi\lambda y\sqrt{1+y'^2} - y'\,\frac{\lambda\; 2\pi y y'}{\sqrt{1+y'^2}} &= c \\ \pi y^2 + \frac{2\pi\lambda y(1+y'^2) - \lambda 2\pi y y'^2}{\sqrt{1+y'^2}} &= c \\ \pi y^2 + \frac{2\pi\lambda y}{\sqrt{1+y'^2}} &= c \end{aligned}$$

Since the curve passes through $(0,0)$, when $y=0, c=0 \quad y^2 = \dfrac{-2\lambda y}{\sqrt{1+y'^2}}$.

$$\begin{aligned} y &= \frac{-2\lambda}{\sqrt{1+y'^2}} \\ \Longrightarrow \quad y^2(1+y'^2) &= 4\lambda^2 \\ \Longrightarrow \quad y'^2 &= \frac{4\lambda^2 - y^2}{y^2} \\ y' &= \frac{\sqrt{4\lambda^2 - y^2}}{y} \\ \int \frac{y\; dy}{\sqrt{4\lambda^2 - y^2}} &= \int dx + k \\ x &= k - \sqrt{4\lambda^2 - y^2} \\ \text{When} \quad x = 0, y = 0 \quad \Longrightarrow \quad k &= 2\lambda \\ x &= 2\lambda - \sqrt{4\lambda^2 - y^2} \\ (x - 2\lambda) &= -\sqrt{4\lambda^2 - y^2} \\ (x-2\lambda)^2 + y^2 &= 4\lambda^2 \end{aligned}$$

The curve is a circle, centred at $(2\lambda, 0)$ and radius 2λ. Hence the solid of revolution is a sphere.

Lecture-5

(i) Problem with Higher order Derivatives

$$
\begin{aligned}
\text{Extremize} \quad I(y) &= \int_{x_1}^{x_2} F(x, y, y', y'')\ dx \\
y(x_1) &= y_1, \quad y(x_2) = y_2 \\
y'(x_1) &= y_1' \quad y'(x_2) = y_1'
\end{aligned}
$$

The Euler - Poisson Equation is

$$
\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0
$$

Example:

Extremize $I(y) = \displaystyle\int_{x_1}^{x_2} (y^2 - (y'')^2)\ dx$
with end conditions:

$$
\begin{aligned}
y(x_1) &= y_1, \quad y(x_2) = y_2 \\
y'(x_1) &= y_1', \quad y'(x_2) = y_2'
\end{aligned}
$$

$$
\frac{\partial F}{\partial y} = 2y, \quad \frac{\partial F}{\partial y'} = 0, \quad \frac{\partial F}{\partial y''} = -2y''
$$

Euler - Poisson Equation :

$$
\begin{aligned}
\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) &= 0 \\
2y - 0 + \frac{d^2}{dx^2}\left(-2y''\right) &= 0 \\
y^{(iv)} - y &= 0
\end{aligned}
$$

$$
\begin{aligned}
y(x_1) &= y_1, \quad y(x_2) = y_2 \\
y'(x_1) &= y_1', \quad y'(x_2) = y_2'
\end{aligned}
$$

(ii) Problems with several unknown functions

Let u and v be the unknown functions which extremize the functional I.

$$
\begin{aligned}
\text{Extremize} \quad I(u, v) &= \int_{x_1}^{x_2} F(x, u, v, u', v')\ dx \\
u(x_1) &= u_1, \quad u(x_2) = u_2 \\
v(x_1) &= v_1, \quad v(x_2) = v_2
\end{aligned}
$$

Euler - Lagrange equations:

$$\begin{aligned} \frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) &= 0 \\ \frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v'}\right) &= 0 \end{aligned}$$

(iii) Problems with more than one independent variables

Let z be the dependent variable and x and y be the independent variables. Extremize $I(z) = \int_{x_1}^{x_2}\int_{y_1}^{y_2} F(x, y, z, z_x, z_y)dy\ dx$

where z is prescribed on the boundary ∂D of the domain D where F is defined.

Euler - Lagrange Equation

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) = 0$$

Example

Find a function Φ whose mean square value of the magnitude of the gradient over a region D is minimum
The problem is

$$\text{Minimize } I(\Phi) = \int\int (\Phi_x^2 + \Phi_y^2)\ dx\ dy$$

where Φ is prescribed on the boundary ∂D of D.

$$\text{Here } F = \Phi_x^2 + \Phi_y^2$$

$$\begin{aligned} \text{Euler Lagrange Eqn: } \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \Phi_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial \Phi_y}\right) &= 0 \\ \frac{\partial}{\partial x}(2\Phi_x) + \frac{\partial}{\partial y}(2\Phi_y) &= 0 \\ \Phi_{xx} + \Phi_{yy} &= 0 \\ \Delta\Phi &= 0 \quad \text{(Laplace Equation)} \\ \Phi\big|_{\partial D} &= \text{ prescribed .} \end{aligned}$$

Lecture-6

The Variational Notation:

When a function changes its value from $y(x)$ to $y(x + \Delta x)$, the rate of change of this defines the derivative $y'(x)$. Whereas in variational calculus the function $y(x)$ is changed to a new function

$y(x) + \epsilon\ \eta(x)$, where ϵ is a constant and $\eta(x)$ is a continuous differentiable function. The change $\epsilon\ \eta(x)$ in $y(x)$ as a function is called the variation of y and is denoted by δy. That is $\delta y = \epsilon\ \eta(x)$. Similarly we have $\delta y' = \epsilon\ \eta'(x)$. In $F = F(x, y, y')$ for a fixed x, change in y from y to $y + \epsilon\ \eta$ makes F to change to $F(x, y + \epsilon\ \eta, y' + \epsilon\ \eta')$. Thus the change in F, denoted by ΔF is given by

$$\Delta F = F(x, y + \epsilon\ \eta, y' + \epsilon\ \eta') - F(x, y, y')$$

Expanding the first term on RHS in Taylors series

$$\begin{aligned}
\Delta F &= F(x, y, y') + \left(\frac{\partial F}{\partial y}\ \eta + \frac{\partial F}{\partial y'}\ \eta' \right) \epsilon + \left[\frac{\partial^2 F}{\partial y^2}\ \eta^2 + 2\frac{\partial^2 F}{\partial y \partial y'} \eta\ \eta' + \frac{\partial^2 F}{\partial y'^2} (\eta')^2 \right] \frac{\epsilon^2}{2!} \\
&\quad + \text{higher order terms of } (\epsilon) - F(x, y, y') \\
&= \frac{\partial F}{\partial y}\ \delta y + \frac{\partial F}{\partial y'}\ \delta y' + \frac{1}{2!} \left[\frac{\partial^2 F}{\partial y^2} (\delta y)^2 + 2\frac{\partial^2 F}{\partial y \partial y'} \delta y\ \delta y' + \frac{\partial^2 F}{\partial y'^2} (\delta y')^2 \right]^2 \\
&\quad + \text{higher order terms}
\end{aligned}$$

$$\begin{aligned}
\text{First variation} \quad \delta F &= \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'}\ \delta y' \\
\text{Second Variation} \quad \delta^2 F &= \frac{1}{2} \left[\frac{\partial^2 F}{\partial y^2} (\delta y)^2 + 2\frac{\partial^2 F}{\partial y \partial y'} \delta y\ \delta y' + \frac{\partial^2 F}{\partial y'^2} (\delta y')^2 \right]
\end{aligned}$$

Variation is analogous to derivative in calculus

Properties:

(1) $\delta(F_1 \pm F_2) = \delta F_1 \pm \delta F_2$

(2) $\delta(F_1 F_2) = F_1 \delta F_2 + F_2 \delta F_1$

(3) $\delta\left(\frac{F_1}{F_2}\right) = \frac{F_2 \delta F_1 - F_1 \delta F_2}{F_2^2}, \quad F_2 \neq 0$

(4) $\delta(F^n) = nF^{n-1} \delta F$

Example

(i) $\delta(y^2) = 2y\delta y$

(ii) $\delta(y'^2) = 2y'\delta y'$

(iii) $\delta(xy) = x\delta y$

(iv) $\delta(x^2) = 0$

Problem:

If $I(y) = \int_{x_1}^{x_2} F(x, y, y')\ dx$, find the variation δI of I.

Solution:

$$
\begin{aligned}
\delta I &= \delta\left(\int_{x_1}^{x_2} F(x,y,y')\ dx\right) \\
&= \int_{x_1}^{x_2} \delta F(x,y,y')\ dx \\
&= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right] dx
\end{aligned}
$$

In the classical calculus, if x_0 an optimizing point for a differentiable function $f(x)$ then $f'(x_0) = 0$. Analogous to this, we have the following result in calculus of variations.

Theorem:

If $y(x)$ is an extremizing function for

$$
\text{Extremize} \quad I(y) = \int_{x_1}^{x_2} F(x,y,y')\ dx, \quad y(x_1) = y_1, \quad y(x_2) = y_2
$$

Then the first variations $\delta I(y) = 0$

Proof:

The first variation of I is given by

$$
\begin{aligned}
\delta I &= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right]\ dx \\
&= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\frac{d}{dx}(\delta y)\right]\ dx
\end{aligned}
$$

Integrating by points on the

$$
\begin{aligned}
\text{second term,} \quad \int \frac{\partial F}{\partial y'}\frac{d}{dx}(\delta y)dx &= \left.\frac{\partial F}{\partial y'}\delta y\right]_{x_1}^{x_2} \overset{\nearrow 0}{} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y\ \delta x \\
\delta I(y) &= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y\ dx \\
&= 0 \quad \text{because of Euler-Lagrange Equation} \\
&\quad\ \text{for an extremizer.} \\
\text{Hence } \delta I(y) &= 0 \quad \text{if} \quad y \quad \text{is an extremizing function.}
\end{aligned}
$$

Hamilton's Principle

Let T be the kinetic energy and V be the potential energy of a particle in motion. Let $L = T - V$ be the kinetic potential or the Lagrangian function.

$$\begin{aligned} \text{Let} \quad A &= \int_{t_1}^{t_2} L\ dt \quad \text{(Action integral)} \\ \delta A &= 0 \quad \text{(Principle of Least Action)} \\ \delta\left(\int_{t_1}^{t_2} L\ dt\right) &= \delta\left(\int_{t_1}^{t_2} T - V\ dt\right) = 0 \end{aligned}$$

Hamilton Principle states that the motion is such that the integral of the difference between kinetic and potential energies is stationary for the true path. Over a sufficiently small time interval the integral is a minimum. That is, nature tends to equalize the kinetic and potential energies over motion. Hence $\delta A = 0$ along truth path.

Second Order Conditions for Extremum.

As in the classical calculus, if f is differentiable and x_0 is a stationary point of f then $f''(x_0) > 0$ implies x_0 is minimum & $f''(x_0) < 0$ implies x_0 is maximum point, we have the following second order conditions for testing extremum in Calculus of variations.

Legendre Test for Extremum

Let I be a functional and y be an extremizer of I then

(i) $\delta I(y) = 0$. (Euler-Lagrange Equation)

(ii) $\delta^2 I(y) > 0 \implies y$ is a minimizing function.

(iii) $\delta^2 I(y) < 0 \implies y$ is a maximizing function.

Lecture-7

Reduction of BVP into Variational Problems

If a variational problem is given, the corresponding Euler-Lagrange equation is a BVP. Now, we ask the question, if a BVP is given, can we find its corresponding variational problem. The answer is yes for a class of BVP. We demonstrate it with the following example:
Reduce the BVP

$$\begin{aligned} y'' - y + x &= 0 \qquad (6) \\ y(0) = y(1) &= 0 \end{aligned}$$

into a variational problem.

Solution:

Multiply both sides of (6) by δy and integrate over $(0,1)$.

$$\int_0^1 y''\delta y\ dx - \int y\ \delta y\ dx + \int x\ \delta y\ dx = 0$$

Integration by parts,

$$\begin{aligned}
y'\delta y\Big]_0^1 - \int_0^1 y'\ \delta y'\ dx - \int y\ \delta y\ dx + \int x\ \delta y\ dx &= 0 \\
\text{But}\quad \delta(y'^2) = 2y'\delta y',\quad \delta y^2 = 2y\ \delta y\quad \delta(xy) &= x\ \delta y \\
-\int_0^1 \frac{1}{2}\delta y'^2 dx - \int_0^1 \frac{1}{2}\delta y^2\ dx + \int \delta xy\ dx &= 0 \\
\int_0^1 \delta(-\frac{1}{2}y'^2 - \frac{1}{2}y^2 + xy)\ dx &= 0 \\
\delta\left(\int_0^1 y'^2 + y^2 - 2xy\ dx\right) &= 0 \\
\text{It is of the form } \delta I(y) &= 0
\end{aligned}$$

Thus the corresponding variational problem is

$$\left.\begin{aligned} \text{Extremize}\quad I(y) &= \int_0^1 y'^2 + y^2 - 2xy\ dx \\ & y(0)=0,\quad y(1)=0 \end{aligned}\right\}\text{V.P}$$

If we find the Euler - Lagrange equation of the above V.P, we have

$$\begin{aligned}
F &= y'^2 - y^2 - 2xy \\
\frac{\partial F}{\partial y} &= 2y - 2x \\
\frac{\partial F}{\partial y'} &= 2y'
\end{aligned}$$

Euler-Lagrange eqn. is given by

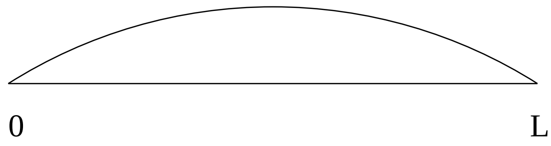
$$2y - 2x - \frac{d}{dx}(2y') = 0$$

$$\left.\begin{aligned} y'' - y + x = 0 \\ y(0) = y(1) = 0 \end{aligned}\right\}\text{which is same as the original BVP}$$

Example 2:

Deflection of a rotating string of Length L.
Consider the boundary value problem

$$\begin{aligned}
\frac{d}{dx}\left(F(x)\frac{dy}{dx}\right) + \rho\omega^2 y + p(x) &= 0 \qquad (7) \\
y(0) = 0\quad y(L) &= 0
\end{aligned}$$



where

$$
\begin{aligned}
y(x) &\quad - \quad \text{displacement of a point from the axis of rotation.}\\
F(x) &\quad - \quad \text{tension.}\\
\rho(x) &\quad - \quad \text{linear mass density.}\\
\omega &\quad - \quad \text{angular velocity of rotation.}\\
p(x) &\quad - \quad \text{intensity of distributed radial load.}
\end{aligned}
$$

We now reduce this BVP into a variational problem as follows:

Multiply (7) by a variation δy and integrate over $(0, L)$ to obtain

$$\int_0^L \frac{d}{dx}(F\frac{dy}{dx})\delta y \; dx + \int_0^L \rho\omega^2 y \; \delta y \; dx + \int_0^L p \; \delta y \; dx = 0$$

Consider the first term and integration by parts gives

$$
\begin{aligned}
\int_0^L \frac{d}{dx}\left(F\,\frac{dy}{dx}\right)\delta y \; dx &= \left(F\frac{dy}{dx}\delta y\right]_0^L \rightarrow 0 - \int_0^L F\frac{dy}{dx}\,\delta\,\frac{dy}{dx}dx\\
\text{But} \quad \delta(y'^2) &= 2y'\delta y', \quad \delta(y^2) = 2y\;\delta y \text{ reduce}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^L \delta\left(-\frac{1}{2}Fy'^2\right) + \rho\omega^2\delta\left(\frac{1}{2}y^2\right) + \delta py \;\; dx &= 0\\
\int_0^L \delta\left(-\frac{1}{2}Fy'^2 + \frac{1}{2}\rho\omega^2 y^2 + py\right) \; dx &= 0\\
\text{That is,} \qquad \delta I &= 0
\end{aligned}
$$

Thus the variational problem is:

$$
\begin{aligned}
\text{Extremize} \quad I(y) &= \int_0^L \left(-Fy'^2 + \rho\omega^2 y^2 + 2py\right) dx\\
y(0) = 0, \qquad & y(L) = 0
\end{aligned}
$$

Example

Reduce the BVP

$$
\begin{aligned}
\frac{d}{dx}\left(x\frac{dy}{dx}\right) + y &= x\\
y(0) = 0, \qquad & y(1) = 1
\end{aligned}
$$

into a variational problem.

Solution:

Multiplying by δy and integrating over $(0,1)$

$$\begin{aligned}
\int_0^1 \left(x\frac{dy}{dx}\right)' \delta y\ dx + \int_0^1 y\ \delta y - \int_0^1 x\delta y &= 0 \\
\left.(x\frac{dy}{dx})\delta y\right]_0^1 \nearrow 0 - \int_0^1 x\frac{dy}{dx}\delta\frac{dy}{dx}dx + \int \delta\left(\frac{1}{2}y^2\right)dx - \int \delta xy\ dx &= 0 \\
\int_0^1 \delta\left(\frac{-xy'^2}{2} + \frac{1}{2}y^2 - xy\right)dx &= 0 \\
\delta\int_0^1 (-xy'^2 + y^2 - 2xy)\ dx &= 0 \\
\delta I &= 0
\end{aligned}$$

$$\begin{aligned}
\text{where}\quad I(y) &= \int_0^1 -xy'^2 + y^2 - 2xy\ dx \\
y(0) = 0, &\qquad y(1) = 1
\end{aligned}$$

Lecture-8

Direct Method to Solve Variational Problems

Rayleigh Ritz Method to find approximate solution

Let $C^1[x_1, x_2]$ be the set of all continuously differentiable functions defined on $[x_1, x_2]$. Consider the variational problem:

$$\text{Min}\ \ I(y) = \int_{x_1}^{x_2} F(x, y, y')\ dx, \quad y(x_1) = y_1\ \&\ y(x_2) = y_2$$

Let $y(x) \in C^1[x_1, x_2]$ be the solution to the V.P.
Let $B = \{\phi_0(x), \phi_1(x),\ ..., \phi_n(x), ...\}$ be basis for the infinite dimensional vectorspace $C^1[x_1, x_2]$.
Let $\bar{y}(x)$ be an approximation of y given by

$$\bar{y}(x) = \sum_{i=0}^{n} c_i\phi_i(x)$$

The basis functions are taken such that the boundary condition $\bar{y}(x_1) = y_1$ and $\bar{y}(x_2) = y_2$ are satisfied.
The problem becomes

$$\begin{aligned}
\text{Minimize } I(y) &= \int_{x_1}^{x_2} F(x, \sum_{i=0}^{\infty} c_i\phi_i(x), \sum_{i=0}^{\infty} c_i\phi_i'(x))dx \\
y(x_1) = y_1 \quad &\& \quad y(x_2) = y_2
\end{aligned}$$

The problem to find an approximate solution $\bar{y}$ in

$$\begin{aligned} \text{Minimize } I(\bar{y}) &= \int_{x_1}^{x_2} F(x, \sum_{i=0}^{n} c_i\phi_i(x), \sum_{i=0}^{n} c_i\phi_i'(x))dx \\ \bar{y}(x_1) = y_1 \quad \& \quad \bar{y}(x_2) = y_2 \end{aligned}$$

Since $\phi_0, \phi_1, ...$ are known basic functions, the only unknown are $c_0, c_1, ..., c_n$, we have

$$\min \ I(\bar{y}) = \min_{c_0, c_1, \cdots, c_n} I(c_0, c_1, \cdots, c_n)$$

Using the classical calculus, we have

$$\frac{\partial I}{\partial c_i} = 0, \quad i = 0, 1, 2, \cdots, n.$$

If we simplify this $n+1$ equation, we need to solve $n+1$ linear equation in $n+1$ unknowing to set $c_0, c_1, \cdots, c_n$.

Example

Find approximate solution to the BVP

$$y'' - y + x = 0, \quad y(0) = y(1) = 0$$

by using Rayleigh-Ritz Method.

Solution:

$$\begin{aligned} I(y) &= \int_0^1 2xy - y^2 - y'^2 \ dx \\ \text{Let} \quad \bar{y}(x) &= c_0 + c_1x + c_2x^2 \text{ be an approximate solution.} \end{aligned}$$

Applying the both condition

$$\begin{aligned} \bar{y}(0) = 0 &\implies c_0 = 0 \\ \bar{y}(1) = 0 &\implies c_1 + c_2 = 0 \quad c_2 = -c_1 \\ \text{Thus } \bar{y}(x) &= c_1 \ x(1-x), \text{where } c_1 \text{ has to be determined} \\ I(c_1) &= \int_0^1 2x\bar{y} - \bar{y}^2 - \bar{y}'^2 \ dx \\ &= \int_0^1 (2c_1(x^2 - x^3) - c_1^2(x - x^2)^2 - c_1(1-x)^2 \ dx \\ &= \frac{1}{6}c_1 - \frac{11}{30}c_1^2 \\ \frac{dI(c_1)}{dc_1} &= 0 \\ \implies \quad \frac{1}{6} - \frac{22}{30}c_1 &= 0 \\ \frac{22}{30}c_1 &= \frac{1}{6} \\ c_1 &= \frac{5}{22} \end{aligned}$$

$$\bar{y}(x) = \frac{5}{22}x(1-x) \text{ is the approximate solution.}$$

$$\text{Exact Solution:} \quad y(x) = x - \frac{e^x - e^{-x}}{e - e^{-1}}$$

x	Approximate solution	Exact solution
0.25	0.043	0.035
0.50	0.057	0.057
0.75	0.43	0.05

References

1. Louis Komzsik: Applied Calculus of variations for Engineers, CRC Press, NewYork, 2009.

2. C. R. Wylie and L.C Barrett: Advanced Engineering Mathematics, McGrawHill Inc, Singapore - 1982.

3. B. S. Grewal: Higher Engineering Mathematics, Khanna Publishers, New Delhi, 2005.

4. Tyn Myint U: Linear Partial Differential Equations for scientists and Engineers Birkhauser, Boston, 2007.

Assignment

Submit ALL starred problems by 25^{th} March 2014.

1. Solve the Euler-Lagrange equation for the functional

$$\int_{1/10}^{1} y'(1+x^2y')dx$$

subject to the end conditions $y(\frac{1}{10}) = 19, y(1) = 1$.

2. Derive Euler-Lagrange equation for the variational problem

$$Extremize\ I(y) = \int_{x_1}^{x_2} F(x,y,y')dx,\ \ y(x_1) = y_1 \ and \ y(x_2) = y_2.$$

Deduce Beltrami identity from it.

*3. Find the curve on which the functional

$$\int_0^1 (y'^2 + 12xy)dx \ with \ y(0) = 0, y(1) = 1$$

has extremum value.

4. Find an extremal for the functional $I(y) = \displaystyle\int_0^{\pi/2} [y'^2 - y^2]dx$ which satisfies the boundary conditions $y(0) = 0$ and $y(\frac{\pi}{2}) = 1$.

5. Show that the Euler-Lagrange equation can also be written in the form

$$F_y - F_{y'x} - F_{y'y}y' - F_{y'y'}y'' = 0.$$

*6. It is required to determine the continuously differentiable function $y(x)$ which minimizes the integral $I(y) = \displaystyle\int_0^1 (1+y'^2)dx$, and satisfies the end conditions $y(0) = 0, y(1) = 1$.

   (a) Obtain the relevant Euler equation, and show that the stationary function is $y = x$.

   (b) With $y(x) = x$ and the special choice $\eta(x) = x(1-x)$ and with the notation $I(\epsilon) = \displaystyle\int_0^1 F(x, y+\epsilon\eta(x), y'+\epsilon\eta'(x))dx$, calculate $I(\epsilon)$ and verify directly that $\frac{dI(\epsilon)}{d\epsilon} = 0$ when $\epsilon = 0$.

7. Find the extremal of the following functionals

   (a) $I(y) = \displaystyle\int_{x_1}^{x_2} \left[y^2 - (y')^2 - 2y\cos hx\right]dx, \quad y(x_1) = y_1 \ \&\ y(x_2) = y_2$

   (b) $I(y) = \displaystyle\int_{x_1}^{x_2} \frac{1+y^2}{y'^2}dx$

   *(c) $I(y) = \displaystyle\int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{x}dx$

(d) $I(y) = \int_0^1 (xy + y^2 - 2y^2y')dx, \quad y(0) = 1, y(1) = 2$

(e) $I(y) = \int_{x_1}^{x_2} (y^2 + y'^2 - 2y\sin x)dx$

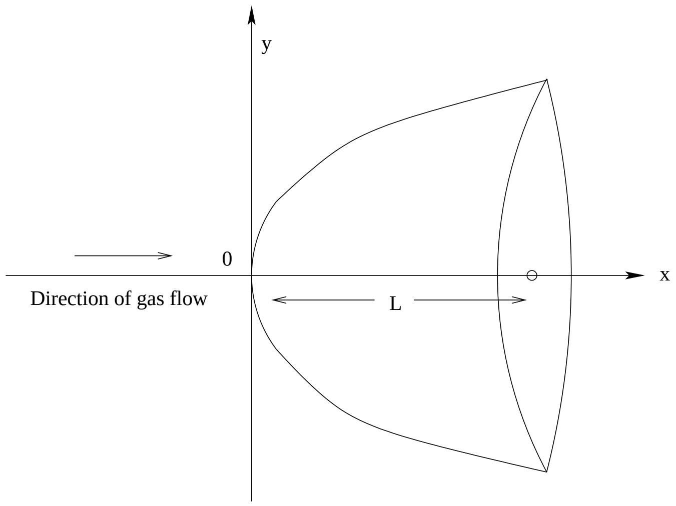
(f) $\int_0^{\pi/2} (y'^2 - y^2 + 2xy)dx, \quad y(0) = 0, y(\frac{\pi}{2}) = 0$

(g) $\int_{x_1}^{x_2} (y^2 + 2xyy')dx; \quad y(x_1) = y_1, y(x_2) = y_2$

*(h) $\int_0^{\pi} (4y\cos x - y^2 + y'^2)dx; \quad y(0) = 0, y(\pi) = 0$

*(i) $I(y) = \int_{x_0}^{x_1} (y^2 + y'^2 + 2ye^x)dx$

8. Determine the shape of solid of revolution moving in a flow of gas with least resistance.



(Hint : The total resistance experienced by the body is $I(y) = 4\pi\rho v^2 \int_0^L yy'^3 dx$ where ρ is the density, v is the velocity of gas relative to the solid).

9. Prove the following facts by using COV:
   (a) The shortest distance between two points in a plane is a **straight line.**
   (b) The curve passing through two points on xy plane which when rotated about $x - axis$ giving a minimum surface area is a **Catenary.**
   (c) The path on which a particle in absence of friction slides from one point to another in the shortest time under the action of gravity is a **Cycloid**(Brachistochrone Problem).

*10. Find the extremal of the functional

$$I(y) = \int_0^{\pi} (y'^2 - y^2)dx, \quad y(0) = 0, \; y(\pi) = 1$$

and subject to the constraint $\int_0^{\pi} y\,dx = 1.$

11. Find the extremal of the isoperimetric problem

$$\text{Extremize} \quad I(y) = \int_1^4 y'^2 dx, \quad y(1) = 3, y(4) = 24$$

subject to $\int_1^4 y\, dx = 36.$

*12. Determine $y(x)$ for which $\int_0^1 x^2 + y'^2 dx$ is stationary subject to $\int_0^1 y^2 dx = 2, \quad y(0) = 0, y(1) = 0.$

13. Find the extremal of $I = \int_0^\pi y'^2 dx$ subject to $\int_0^\pi y^2 dx = 1$ and satisfying $y(0) = y(\pi) = 0.$

*14. Given $F(x, y, y') = (y')^2 + xy$. Compute ΔF and δF for $x = x_0$, $y = x^2$ and $\delta y = \epsilon x^n$.

15. Find the extremals of the isopermetric problem

$$I(y) = \int_{x_0}^{x_1} y'^2 dx$$

given that $\int_{x_0}^{x_1} y dx =$ constant.

16. Prove the following facts by using COV:

*(a) The geodesics on a sphere of radius a are its great circles.

(b) The sphere is the solid figure of revolution which, for a given surface area has maximum volume.

17. If y is an extremizing function for

$$I(y) = \int_{x_1}^{x_2} F(x, y, y'), y(x_1) = y_1, and\ y(x_2) = y_2$$

then show that of $\delta I = 0$ for the function y.

*18. Find $y(x)$ for which

$$\delta \left\{ \int_{x_0}^{x_1} (\frac{y'^2}{x^3}) dx \right\} = 0$$

and $y(x_1) = y_1$ and $y(x_2) = y_2$.

19. Write down the Euler-Lagrange equation for the following extremization problems

(i) Extremize $I(u, v) = \int_D \int F(x, y, u, v, u_x, u_y, v_x, v_y) dx\ dy$ where x, y are independent variables and u, v are dependent variables. D is a domain in xy plane and u and v are prescribed on the boundary of D.

(ii) Extremize $I(y) = \int_{x_0}^{x_1} F(x, y, y^{(1)}, y^{(2)}, ..., y^{(m)} dx$

$$y(x_0) = y_0,\ y(x_1) = y_1$$

$$y'(x_0) = y_0', \; y'(x_1) = y_1'$$

$$\dots\dots\dots\dots\dots$$

$$y^{(m-1)}(x_0) = y_0^{(m-1)}, \; y^{(m-1)}(x_1) = y_1^{(m-1)}$$

(iii) Max or Min $I(y) = \int_{x_1}^{x_2} F(x, y, y')dx$ where y is prescribed at the end points $y(x_1) = y_1$, $y(x_2) = y_2$, and y is also to satisfy the integral constraint condition $J(y) = \int_{x_1}^{x_2} G(x, y, y')dx = k$, where k is a prescribed constant.

*20. Show that the extremals of the problem
Extremize $I(y) = \int_{x_1}^{x_2} [p(x)y'^2 - q(x)y^2]dx$
where $y(x_1)$ and $y(x_2)$ are prescribed and y satisfies a constraint $\int_{x_1}^{x_2} r(x)y^2(x)dx = 1$, are solutions of the differential equation $\frac{d}{dx}(p\frac{dy}{dx}) + (q + \lambda r)y = 0$
where λ is a constant.

*21. Reduce the BVP

$$\frac{d}{dx}(x\frac{dy}{dx}) + y = x, \; y(0) = 0, \; y(1) = 1$$

into a variational problem and use Rayleigh-Ritz method to obtain an approximate solution in the form

$$y(x) \approx x + x(1 - x)(c_1 + c_2 x)$$

22. (Principle of least Action) A particle under the influence of a gravitational field moves on a path along which the kinetic energy is minimal. Using calculus of variation prove that the trajectory is parabolic.
(Hint: Minimize $I = \int \frac{1}{2}mv^2 dt = \int \frac{1}{2}mvds = \int \sqrt{u^2 - 2gy}\sqrt{1 + y'^2}dx$)
where u is the initial speed.

23. Show that the curve which extremizes the functional $I(y) = \int_0^{\pi/4} (y''^2 - y^2 + x^2)dx$ under the conditions $y(0) = 0, y'(0) = 1, y(\pi/4) = y'(\pi/4) = \frac{1}{\sqrt{2}}$ is $y = \sin x$.

24. Find a function $y(x)$ such that $\int_0^{\pi} y^2 dx = 1$ which makes $\int_0^{\pi} (y'')^2 dx$ a minimum if $y(0) = 0 = y(\pi), \quad y''(0) = 0 = y''(\pi)$.

*25. Find the extremals of the following functional

$$I(y) = \int_{x_1}^{x_2} 2xy + (y''')^2 dx$$

26. Find the extremals of the functional

$$I(u, v) = \int_{x_0}^{x_1} 2uv - 2u^2 + u'^2 - v'^2 dx$$

where u and v are prescribed at the end points.

27. Find a function $y(x)$ such that $\int_0^{\pi} y^2 dx = 1$ which makes $\int_0^{\pi} y''^2 dx$ a minimum if $y(0) = 0 = y(\pi), \quad y''(0) = 0 = y''(\pi)$

*28. Show that the functional $\int_0^{\pi/2} 2xy \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 dt$ such that $x(0) = 0,\ x(\pi/2) = -1,\ y(0) = 0,\ y(\pi/2) = 1$ is stationary for $x = -\sin t,\ y = \sin t$.

*29. Explain Rayleigh - Ritz method to find an approximate solution of the variational problem

$$\text{Extremize}\, I(y) = \int_{t_0}^{t_1} F(x, y, y')dx$$

with prescribed end conditions $y(x_1) = y_1 \quad \& \quad y(x_2) = y_2$.

30. Solve the BVP $y'' + y + x = 0, \quad y(0) = y(1) = 0$ by Rayleigh - Ritz method.

31. Use Rayleigh - Ritz method to find an approximate solution of the problem $y'' - y + 4xe^x = 0, \quad y'(0) - y(0) = 1, \quad y'(1) + y(1) = -e$.

*****END*****