Classification of Partial Differential Equations and Canonical Forms

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22 December 2014

1 Second-Order Partial Differential Equations

The most general case of second-order linear partial differential equation (PDE) in two independent variables is given by

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \tag{1}$$

where the coefficients A, B, and C are functions of x and y and do not vanish simultaneously, because in that case the second-order PDE degenerates to one of first order. Further, the coefficients D, E, and F are also assumed to be functions of x and y. We shall assume that the function $u(x,y)$ and the coefficients are twice continuously differentiable in some domain Ω.

The classification of second-order PDE depends on the form of the leading part of the equation consisting of the second order terms. So, for simplicity of notation, we combine the lower order terms and rewrite the above equation in the following form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} = \Phi\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right) \tag{2a}$$

or using the short-hand notations for partial derivatives,

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} = \Phi(x,y,u,u_x,u_y) \tag{2b}$$

As we shall see, there are fundamentally three types of PDEs – *hyperbolic*, *parabolic*, and *elliptic* PDEs. From the physical point of view, these PDEs respectively represents the wave propagation, the time-dependent diffusion processes, and the steady state or equilibrium processes. Thus, hyperbolic equations model the transport of some physical quantity, such as fluids or waves. Parabolic problems describe evolutionary phenomena that lead to a steady state described by an elliptic equation. And elliptic equations are associated to a special state of a system, in principle corresponding to the minimum of the energy.

Mathematically, these classification of second-order PDEs is based upon the possibility of reducing equation (2) by coordinate transformation to *canonical* or *standard form* at a point. It may be noted that, for the purposes of classification, it is not necessary to restrict consideration to linear equations. It is applicable to quasilinear second-order PDE as well. A quasilinear second-order PDE is linear in the second derivatives only.

The type of second-order PDE (2) at a point (x_0, y_0) depends on the sign of the discriminant defined as

$$\Delta(x_0, y_0) \equiv \begin{vmatrix} B & 2A \\ 2C & B \end{vmatrix} = B(x_0, y_0)^2 - 4A(x_0, y_0)C(x_0, y_0) \tag{3}$$

The classification of second-order linear PDEs is given by the following: If $\Delta(x_0, y_0) > 0$, the equation is hyperbolic, $\Delta(x_0, y_0) = 0$ the equation is parabolic, and $\Delta(x_0, y_0) < 0$ the equation is elliptic. It should be remarked here that a given PDE may be of one type at a specific point, and of another type at some other point. For example, the Tricomi equation

$$\frac{\partial^2 u}{\partial x^2} + x\frac{\partial^2 u}{\partial y^2} = 0$$

is hyperbolic in the left half-plane $x < 0$, parabolic for $x = 0$, and elliptic in the right half-plane $x > 0$, since $\Delta = -4x$. A PDE is hyperbolic (or parabolic or elliptic) in a region Ω if the PDE is hyperbolic (or parabolic or elliptic) at each point of Ω.

The terminology hyperbolic, parabolic, and elliptic chosen to classify PDEs reflects the analogy between the form of the discriminant, $B^2 - 4AC$, for PDEs and the form of the discriminant, $B^2 - 4AC$, which classifies conic sections given by

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

The type of the curve represented by the above conic section depends on the sign of the discriminant, $\Delta \equiv B^2 - 4AC$. If $\Delta > 0$, the curve is a hyperbola, $\Delta = 0$ the curve is an parabola, and $\Delta < 0$ the equation is a ellipse. The analogy of the classification of PDEs is obvious. There is no other significance to the terminology and thus the terms hyperbolic, parabolic, and elliptic are simply three convenient names to classify PDEs.

In order to illustrate the significance of the discriminant Δ and thus the classification of the PDE (2), we try to reduce the given equation (2) to a canonical form. To do this, we transform the independent variables x and y to the new independent variables ξ and η through the change of variables

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y) \tag{4}$$

where both ξ and η are twice continuously differentiable and that the Jacobian

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0 \tag{5}$$

in the region under consideration. The nonvanishing of the Jacobian of the transformation ensure that a one-to-one transformation exists between the new and old variables. This simply means that the new independent variables can serve as new coordinate variables without any ambiguity. Now, define $w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$. Then $u(x, y) = w(\xi(x, y), \eta(x, y))$ and, apply the chain rule to compute the terms of the equation (2) in terms of ξ and η as follows:

$$\begin{aligned}
u_x &= w_\xi \xi_x + w_\eta \eta_x \\
u_y &= w_\xi \xi_y + w_\eta \eta_y \\
u_{xx} &= w_{\xi\xi}\xi_x^2 + 2w_{\xi\eta}\xi_x\eta_x + w_{\eta\eta}\eta_x^2 + w_\xi \xi_{xx} + w_\eta \eta_{xx} \\
u_{yy} &= w_{\xi\xi}\xi_y^2 + 2w_{\xi\eta}\xi_y\eta_y + w_{\eta\eta}\eta_y^2 + w_\xi \xi_{yy} + w_\eta \eta_{yy} \\
u_{xy} &= w_{\xi\xi}\xi_x\xi_y + w_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + w_{\eta\eta}\eta_x\eta_y + w_\xi \xi_{xy} + w_\eta \eta_{xy}
\end{aligned} \tag{6}$$

Substituting these expressions into equation (2) we obtain the transformed PDE as

$$aw_{\xi\xi} + bw_{\xi\eta} + cw_{\eta\eta} = \phi\left(\xi, \eta, w, w_\xi, w_\eta\right) \tag{7}$$

where Φ becomes ϕ and the new coefficients of the higher order terms a, b, and c are expressed via the original coefficients and the change of variables formulas as follows:

$$\begin{aligned}
a &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
b &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
c &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2
\end{aligned} \tag{8}$$

At this stage the form of the PDE (7) is no simpler than that of the original PDE (2), but this is to be expected because so far the choice of the new variable ξ and η has been arbitrary. However, before showing how to choose the new coordinate variables, observe that equation (8) can be written in matrix form as

$$\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix} \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix}^T$$

Recalling that the determinant of the product of matrices is equal to the product of the determinants of matrices and that the determinant of a transpose of a matrix is equal to the determinant of that matrix, we get

$$\begin{vmatrix} a & b/2 \\ b/2 & c \end{vmatrix} = \begin{vmatrix} A & B/2 \\ B/2 & C \end{vmatrix} J^2$$

where J is the Jacobian of the change of variables given by (5). Expanding the determinant and multiplying by the factor, -4, to obtain

$$b^2 - 4ac = J^2(B^2 - 4AC) \qquad \Longrightarrow \qquad \delta = J^2\Delta \tag{9}$$

where $\delta = b^2 - 4ac$ is the discriminant of the equation (7). This shows that the discriminant of the transformed equation (7) has the same sign as the discriminant of the original equation (2) and therefore it is clear that any real nonsingular ($J \neq 0$) transformation does not change the type of PDE. Note that the discriminant involves only the coefficients of second-order derivatives of the corresponding PDE.

1.1 Canonical forms

Let us now try to construct transformations, which will make one, or possibly two of the coefficients of the leading second order terms of equation (7) vanish, thus reducing the equation to a simpler form called *canonical from*. For convenience, we reproduce below the original PDE

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} = \Phi(x,y,u,u_x,u_y) \tag{2}$$

and the corresponding transformed PDE

$$a(\xi,\eta)w_{\xi\xi} + b(\xi,\eta)w_{\xi\eta} + c(\xi,\eta)w_{\eta\eta} = \phi\left(\xi,\eta,w,w_\xi,w_\eta\right) \tag{7}$$

We again mention here that for the PDE (2) (or (7)) to remain a second-order PDE, the coefficients A, B, and C (or a, b, and c) do not vanish simultaneously.

By definition, a PDE is hyperbolic if the discriminant $\Delta = B^2 - 4AC > 0$. Since the sign of discriminant is invariant under the change of coordinates (see equation (9)), it follows that for a hyperbolic PDE, we should have $b^2 - 4ac > 0$. The simplest case of satisfying this condition is $a = c = 0$. So, if we try to chose the new variables ξ and η such that the coefficients a and c vanish, we get the following canonical form of hyperbolic equation:

$$w_{\xi\eta} = \psi\left(\xi,\eta,w,w_\xi,w_\eta\right) \tag{10a}$$

where $\psi = \phi/b$. This form is called the *first canonical form* of the hyperbolic equation. We also have another simple case for which $b^2 - 4ac > 0$ condition is satisfied. This is the case when $b = 0$ and $c = -a$. In this case (9) reduces to

$$w_{\alpha\alpha} - w_{\beta\beta} = \psi\left(\alpha,\beta,w,w_\alpha,w_\beta\right) \tag{10b}$$

which is the *second canonical form* of the hyperbolic equation, where $\psi = \phi/a$.

By definition, a PDE is parabolic if the discriminant $\Delta = B^2 - 4AC = 0$. It follows that for a parabolic PDE, we should have $b^2 - 4ac = 0$. The simplest case of satisfying this condition is $c\,(\text{or } a) = 0$. In this case another necessary requirement $b = 0$ will follow automatically (since $b^2 - 4ac = 0$). So, if we try to chose the new variables ξ and η such that the coefficients b and c vanish, we get the following canonical form of parabolic equation:

$$w_{\xi\xi} = \psi\left(\xi,\eta,w,w_\xi,w_\eta\right) \tag{11}$$

where $\psi = \phi/a$.

By definition, a PDE is elliptic if the discriminant $\Delta = B^2 - 4AC < 0$. It follows that for a elliptic PDE, we should have $b^2 - 4ac < 0$. The simplest case of satisfying this condition is $b = 0$ and $c = a$. So, if we try to chose the new variables ξ and η such that b vanishes and $c = a$, we get the following canonical form of elliptic equation:

$$w_{\xi\xi} + w_{\eta\eta} = \psi\left(\xi,\eta,w,w_\xi,w_\eta\right) \tag{12}$$

where $\psi = \phi/a$.

In summary, equation (7) can be reduced to a canonical form if the coordinate transformation $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$ can be selected such that:

- $a = c = 0$ corresponds to the first canonical form of hyperbolic PDE given by

$$w_{\xi\eta} = \psi\left(\xi, \eta, w, w_\xi, w_\eta\right) \tag{10a}$$

- $b = 0$, $c = -a$ corresponds to the second canonical form of hyperbolic PDE given by

$$w_{\alpha\alpha} - w_{\beta\beta} = \psi\left(\alpha, \beta, w, w_\alpha, w_\beta\right) \tag{10b}$$

- $b = c = 0$ corresponds to the canonical form of parabolic PDE given by

$$w_{\xi\xi} = \psi\left(\xi, \eta, w, w_\xi, w_\eta\right) \tag{11}$$

- $b = 0$, $c = a$ corresponds to the canonical form of elliptic PDE given by

$$w_{\xi\xi} + w_{\eta\eta} = \psi\left(\xi, \eta, w, w_\xi, w_\eta\right) \tag{12}$$

We will now examine the kind of transformation required to reduce the PDE to its canonical form.

1.2 Hyperbolic equations

For a hyperbolic PDE the discriminant $\Delta(= B^2 - 4AC) > 0$. In this case, we have seen that, to reduce this PDE to canonical form we need to choose the new variables ξ and η such that the coefficients a and c vanish in (7). Thus, from (8), we have

$$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 \tag{13a}$$
$$c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0 \tag{13b}$$

Dividing equation (13a) and (13b) throughout by ξ_y^2 and η_y^2 respectively to obtain

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0 \tag{14a}$$
$$A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0 \tag{14b}$$

Equation (14a) is a quadratic equation for (ξ_x/ξ_y) whose roots are given by

$$\mu_1(x,y) = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$
$$\mu_2(x,y) = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$$

The roots of the equation (14b) can also be found in an identical manner, so as only two distinct roots are possible between the two equations (14a) and (14b). Hence, we may consider μ_1 as the root of (14a) and μ_2 as that of (14b). That is,

$$\mu_1(x,y) = \frac{\xi_x}{\xi_y} = \frac{-B+\sqrt{B^2-4AC}}{2A} \tag{15a}$$

$$\mu_2(x,y) = \frac{\eta_x}{\eta_y} = \frac{-B-\sqrt{B^2-4AC}}{2A} \tag{15b}$$

The above equations lead to the following two first-order differential equations

$$\xi_x - \mu_1(x,y)\xi_y = 0 \tag{16a}$$

$$\eta_x - \mu_2(x,y)\eta_y = 0 \tag{16b}$$

These are the equations that define the new coordinate variables ξ and η that are necessary to make $a = c = 0$ in (7).

Along the coordinate line $\xi(x,y) =$ constant, we have the total derivative of ξ, $d\xi = 0$. It follows that

$$d\xi = \xi_x dx + \xi_y dy = 0$$

and hence, the slope of such curves is given by

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}$$

We also have a similar result along coordinate line $\eta(x,y) =$ constant, i.e.,

$$\frac{dy}{dx} = -\frac{\eta_x}{\eta_y}$$

Using these results, equation (14) can be written as

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0 \tag{17}$$

This is called the characteristic polynomial of the PDE (2) and its roots are given by

$$\frac{dy}{dx} = \frac{B+\sqrt{B^2-4AC}}{2A} = \lambda_1(x,y) \tag{18a}$$

$$\frac{dy}{dx} = \frac{B-\sqrt{B^2-4AC}}{2A} = \lambda_2(x,y) \tag{18b}$$

The required variables ξ and η are determined by the respective solutions of the two ordinary differential equations (18a) and (18b), known as the characteristic equations of the PDE (2). They are ordinary differential equations for families of curves in the xy-plane along which $\xi =$ constant and $\eta =$ constant. Clearly, these families of curves depend on the coefficients A, B, and C in the original PDE (2).

Integration of equation (18a) leads to the family of curvilinear coordinates $\xi(x,y) = c_1$ while the integration of (18b) gives another family of curvilinear coordinates $\eta(x,y) = c_2$, where c_1 and c_2 are arbitrary constants of integration. These two families of curvilinear coordinates $\xi(x,y) = c_1$ and $\eta(x,y) = c_2$ are called characteristic curves of the hyperbolic equation (2) or, more simply, the characteristics of the equation. Hence, second-order hyperbolic equations have two families of characteristic curves. The fact that $\Delta > 0$ means that the characteristic are real curves in xy-plane.

If the coefficients A, B, and C are constants, it is easy to integrate equations (18a) and (18b) to obtain the expressions for change of variables formulas for reducing a hyperbolic PDE to the canonical form. Thus, integration of (18) produces

$$y = \frac{B+\sqrt{B^2-4AC}}{2A}x + c_1 \qquad \text{and} \qquad y = \frac{B-\sqrt{B^2-4AC}}{2A}x + c_2 \tag{19a}$$

or

$$y - \frac{B+\sqrt{B^2-4AC}}{2A}x = c_1 \qquad \text{and} \qquad y - \frac{B-\sqrt{B^2-4AC}}{2A}x = c_2 \tag{19b}$$

Thus, when the coefficients A, B, and C are constants, the two families of characteristic curves associated with PDE reduces to two distinct families of parallel straight lines. Since the families of curves $\xi =$ constant and $\eta =$ constant are the characteristic curves, the change of variables are given by the following equations:

$$\xi = y - \frac{B+\sqrt{B^2-4AC}}{2A}x = y - \lambda_1 x \tag{20}$$

$$\eta = y - \frac{B-\sqrt{B^2-4AC}}{2A}x = y - \lambda_2 x \tag{21}$$

The first canonical form of the hyperbolic is:

$$w_{\xi\eta} = \psi\left(\xi, \eta, w, w_\xi, w_\eta\right) \tag{22}$$

where $\psi = \phi/b$ and b is calculated from (8)

$$\begin{aligned} b &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\ &= 2A\left(\frac{B^2-(B^2-4AC)}{4A^2}\right) + B\left(-\frac{B}{2A} - \frac{B}{2A}\right) + 2C \\ &= 4C - \frac{B^2}{A} = -\frac{\Delta}{A} \end{aligned} \tag{23}$$

Each of the families $\xi(x,y) =$ constant and $\eta(x,y) =$ constant forms an envelop of the domain of the xy-plane in which the PDE is hyperbolic.

The transformation $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$ can be regarded as a mapping from the xy-plane to the $\xi\eta$-plane, and the curves along which ξ and η are constant in the xy-plane become coordinates lines in the $\xi\eta$-plane. Since these are precisely the characteristic curves, we conclude that when a hyperbolic PDE is in canonical form, coordinate lines are characteristic curves for the PDE. In other words, characteristic curves of a hyperbolic PDE are those curves to which the PDE must be referred as coordinate curves in order that it take on canonical form.

We now determine the Jacobian of transformation defined by (20) and (21). We have

$$J = \begin{vmatrix} -\lambda_1 & 1 \\ -\lambda_2 & 1 \end{vmatrix} = \lambda_2 - \lambda_1$$

We know that $\lambda_1 = \lambda_2$ only if $B^2 - 4AC = 0$. However, for an hyperbolic PDE, $B^2 - 4AC \neq 0$. Hence Jacobian is nonsingular for the given transformation. A consequence of $\lambda_1 \neq \lambda_2$ is that at no point can the particular curves from each family share a common tangent line.

It is easy to show that the hyperbolic PDE has a second canonical form. The following linear change of variables

$$\alpha = \xi + \eta, \qquad \beta = \xi - \eta$$

converts (22) into

$$w_{\alpha\alpha} - w_{\beta\beta} = \psi\left(\alpha, \beta, w, w_\alpha, w_\beta\right) \tag{24}$$

which is the *second canonical form* of the hyperbolic equations.

For PDE with constant coefficients, the required transformation is given by

$$\begin{aligned} \alpha &= \xi + \eta = (y - \lambda_1 x) + (y - \lambda_2 x) \\ &= 2y - (\lambda_1 + \lambda_2)x \\ \beta &= \xi - \eta = (y - \lambda_1 x) - (y - \lambda_2 x) \\ &= (\lambda_2 + \lambda_1)x \end{aligned}$$

Example 1

Show that the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

is hyperbolic, find an equivalent canonical form, and then obtain the general solution.

Solution To interpret the results for (2) that involve the independent variables x and y in terms of the wave equation $u_{tt} - c^2 u_{xx} = 0$, where the independent variables are t and x, it will be necessary to replace x and y in (2) and (6) by t and x. It follows that the wave equation is a constant coefficient equation with

$$A = 1, \qquad B = 0, \qquad C = -c^2$$

We calculate the discriminant, $\Delta = 4c^2 > 0$, and therefore the PDE is hyperbolic. The roots of the characteristic polynomial are given by

$$\lambda_1 = \frac{B + \sqrt{\Delta}}{2A} = c \qquad \text{and} \qquad \lambda_2 = \frac{B - \sqrt{\Delta}}{2A} = -c$$

Therefore, from the characteristic equations (18a) and (18b), we have

$$\frac{dx}{dt} = c, \qquad \frac{dx}{dt} = -c$$

Integrating the above two ODEs to obtain the characteristics of the wave equation

$$x = ct + k_1, \qquad x = -ct + k_2$$

where k_1 and k_2 are the constants of integration. We see that the two families of characteristics for the wave equation are given by $x - ct =$ constant and $x + ct =$ constant. It follows, then, that the transformation

$$\xi = x - ct, \qquad \eta = x + ct$$

reduces the wave equation to canonical form. We have,

$$a = 0, \qquad c = 0, \qquad b = -\frac{\Delta}{A} = -4c^2$$

So in terms of characteristic variables, the wave equation reduces to the following canonical form

$$w_{\xi\eta} = 0$$

For the wave equation the characteristics are found to be straight lines with negative and positive slopes as shown in Fig. 1. The characteristics form a natural set of coordinates for the hyperbolic equation.

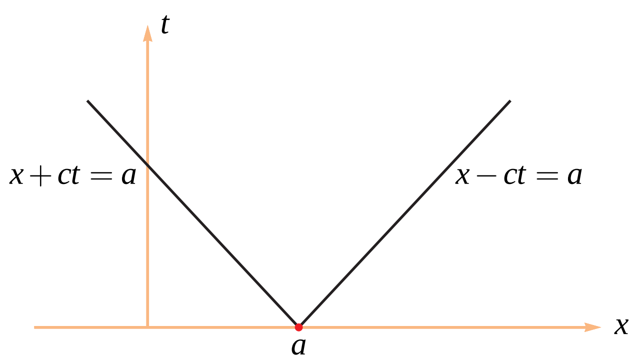

Figure 1: The pair of characteristic curves for wave equation.

The canonical forms are simple because they can be solved directly by integrating twice. For example, integrating with respect to ξ gives

$$w_\eta = \int 0 \, d\xi = h(\eta)$$

where the 'constant of integration' h is an arbitrary function of η. Next, integrating with respect to η to obtain

$$w(\xi, \eta) = \int h(\eta) \, d\eta + f(\xi) = f(\xi) + g(\eta)$$

where f and g are arbitrary twice differentiable functions and g is just the integral of the arbitrary function h. The form of the general solutions of the wave equation in terms of its original variable x and t are then given by

$$u = f(x-ct) + g(x+ct)$$

Note that f is constant on "wavefronts" $x = ct + \xi$ that travel towards right, whereas g is constant on wavefronts $x = -ct + \eta$ that travel towards left. Thus, any general solution can be expressed as the sum of two waves, one travelling to the right with constant velocity c and the other travelling to the left with the same velocity c. This is one of the few cases where the general solution of a PDE can be found.

As mentioned earlier, hyperbolic PDE has an alternate canonical form with the following linear change of variables $\alpha = \xi + \eta$ and $\beta = \xi - \eta$, given by

$$w_{\alpha\alpha} - w_{\beta\beta} = 0$$

Example 2

In steady or unsteady transonic flow around wings and airfoils with thickness to chord ratios of a few percent, we can generally consider that the flow is predominantly directed along the chordwise direction, taken as the x-direction. In this case, the velocities in the transverse direction can be neglected and the potential equation reduces to the so-called small disturbance potential equation:

$$\left(1-M_\infty^2\right)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

Historically, this was the form of equation used by Murman and Cole (1961) to obtain the first numerical solution for a transonic flow around an airfoil with shocks.

Show that, depending on the Mach number, the small disturbance potential equation is elliptic, parabolic, or hyperbolic. Find the characteristic variables for the hyperbolic case and hence write the equation in canonical form.

Solution The given equation is of the form (2) where

$$A = 1-M_\infty^2, \qquad B = 0, \qquad C = 1$$

The discriminant, $\Delta = B^2 - 4AC = -4(1-M_\infty^2)$. Therefore, the PDE is hyperbolic for $M > 1$, elliptic for $M < 1$, and parabolic for $M = 1$ (along the sonic line). For the case of supersonic flow ($M > 1$), the roots of the characteristic polynomial are given by

$$\lambda_1 = \frac{B+\sqrt{\Delta}}{2A} = \frac{\sqrt{4(M_\infty^2-1)}}{2(1-M_\infty^2)} = -\frac{1}{\sqrt{M_\infty^2-1}}$$
$$\lambda_2 = \frac{B-\sqrt{\Delta}}{2A} = -\frac{\sqrt{4(M_\infty^2-1)}}{2(1-M_\infty^2)} = \frac{1}{\sqrt{M_\infty^2-1}}$$

Therefore, from the characteristic equations (18a) and (18b), we have

$$\frac{dy}{dx} = \frac{1}{\sqrt{M_\infty^2 - 1}}, \qquad -\frac{1}{\sqrt{M_\infty^2 - 1}}$$

Integrating the above two ODEs to obtain the characteristics of the wave equation

$$y = \frac{x}{\sqrt{M_\infty^2 - 1}} + c_1, \qquad y = -\frac{x}{\sqrt{M_\infty^2 - 1}} + c_2$$

where c_1 and c_2 are the constants of integration. We see that the two families of characteristics for the wave equation are given by $y - x/\sqrt{M_\infty^2 - 1} =$ constant and $y + x/\sqrt{M_\infty^2 - 1} =$ constant. It follows, then, that the transformation

$$\xi = y - \frac{x}{\sqrt{M_\infty^2 - 1}}, \qquad \eta = y + \frac{x}{\sqrt{M_\infty^2 - 1}}$$

reduces the wave equation to canonical form. From the relations (6), we have

$$\begin{aligned}
\phi_{xx} &= w_{\xi\xi}\xi_x^2 + 2w_{\xi\eta}\xi_x\eta_x + w_{\eta\eta}\eta_x^2 + w_\xi\xi_{xx} + w_\eta\eta_{xx} \\
&= \frac{1}{M_\infty^2 - 1}w_{\xi\xi} - \frac{2}{M_\infty^2 - 1}w_{\xi\eta} + \frac{1}{M_\infty^2 - 1}w_{\eta\eta} \\
\phi_{yy} &= w_{\xi\xi}\xi_y^2 + 2w_{\xi\eta}\xi_y\eta_y + w_{\eta\eta}\eta_y^2 + w_\xi\xi_{yy} + w_\eta\eta_{yy} \\
&= w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}
\end{aligned}$$

Substituting these relations in the given PDE to obtain

$$w_{\xi\eta} = 0$$

This is the canonical form of the given hyperbolic PDE. Here $\xi =$ const. and $\eta =$ const. lines represent two families of straight lines with slopes, $\pm 1/\sqrt{M_\infty^2 - 1}$.

1.3 Parabolic equations

For a parabolic PDE the discriminant $\Delta = B^2 - 4AC = 0$. In this case, we have seen that, to reduce this PDE to canonical form we need to choose the new variables ξ and η such that the coefficients a and b vanish in (7). Thus, from (8), we have

$$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$

Dividing the above equation throughout by ξ_y^2 to obtain

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0 \tag{25}$$

As the total derivative of ξ along the coordinate line $\xi(x,y) =$ const., $d\xi = 0$. It follows that

$$d\xi = \xi_x dx + \xi_y dy = 0$$

and hence, the slope of such curves is given by

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}$$

Using this result, equation (25) can be written as

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0 \tag{26}$$

This is called the characteristic polynomial of the PDE (2). Since $B^2 - 4AC = 0$ in this case, the characteristic polynomial (25) has only one root, given by

$$\frac{dy}{dx} = \frac{B}{2A} = \lambda(x,y) \tag{27}$$

Hence we see that for a parabolic PDE there is only one family of real characteristic curves. The required variables ξ is determined by the ordinary differential equation (27), known as the characteristic equations of the PDE (2). This is an ordinary differential equation for families of curves in the xy-plane along which $\xi =$ constant.

To determine the second transformation variable η, we set $b = 0$ in (8) so that

$$\begin{aligned} 2A\xi_x\eta_x + B\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y &= 0 \\ 2A\frac{\xi_x}{\xi_y}\eta_x + B\left(\frac{\xi_x}{\xi_y}\eta_y + \eta_x\right) + 2C\eta_y &= 0 \\ 2A\left(-\frac{B}{2A}\right)\eta_x + B\left[\left(-\frac{B}{2A}\right)\eta_y + \eta_x\right] + 2C\eta_y &= 0 \\ -B\eta_x - \frac{B^2}{2A}\eta_y + B\eta_x + 2C\eta_y &= 0 \\ \left(B^2 - 4AC\right)\eta_y &= 0 \end{aligned}$$

Since $B^2 - 4AC = 0$ for a parabolic PDE, η_y could be an arbitrary function of (x,y) and consequently the transformation variable η can be chosen arbitrarily, as long as the change of coordinates formulas define a non-degenerate transformation.

If the coefficients A, B, and C are constants, it is easy to integrate equation (27) to obtain the expressions for change of variable formulas for reducing a parabolic PDE to the canonical form. Thus, integration of (27) produces

$$y = \frac{B}{2A}x + c_1 \tag{28a}$$

or

$$y - \frac{B}{2A}x = c_1 \tag{28b}$$

Since the families of curves $\xi =$ constant are the characteristic curves, the change of variables are given by the following equations:

$$\xi = y - \frac{B}{2A}x \tag{29}$$

$$\eta = x \tag{30}$$

where we have set $\eta = x$. The Jacobian of this transformation is

$$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} -B/2A & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$$

Now, we have from (8)

$$\begin{aligned} b &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\ &= 2A\left(-\frac{B}{2A}\right) + B + 0 = 0 \end{aligned}$$

In these new coordinate variables given by (29) and (30), equation (7) reduces to following canonical form:

$$w_{\eta\eta} = \psi\left(\xi, \eta, w, w_\xi, w_\eta\right) \tag{31}$$

where $\psi = \phi/c$. As the choice of η is arbitrary, the form taken by ψ will depend on the choice of η. We have from (8)

$$c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = A \tag{32}$$

Equation (7) may also assume the form

$$w_{\xi\xi} = \psi\left(\xi, \eta, w, w_\xi, w_\eta\right) \tag{33}$$

if we choose $c = 0$ instead of $a = 0$.

Example 3

Show that the one-dimensional heat equation

$$\alpha\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

is parabolic, choose the appropriate characteristic variables, and write the equation in equivalent canonical form.

Solution It follows that the heat equation is a constant coefficient equation with

$$A = \alpha, \qquad B = 0, \qquad C = 0$$

We calculate the discriminant, $\Delta = 0$, and therefore the PDE is parabolic. The single root of the characteristic polynomial is given by

$$\lambda = B/2A = 0$$

Therefore, from the characteristic equation (27), we have

$$\frac{dt}{dx} = 0$$

Integrating the above ODE to obtain the characteristics of the wave equation

$$t = k$$

where k is the constant of integration. Here $t = k$ lines represents the characteristics. Since the families of curves $\xi =$ constant are the characteristic curves, the change of variables are given by the following equations:

$$\xi = t, \qquad \eta = x$$

where we have set $\eta = x$. This shows that the given PDE is already expressed in canonical form and thus no change of variable is needed to simplify the structure. Further, we have from (6)

$$u_t = w_\xi \xi_t + w_\eta \eta_t = w_\xi$$

and $c = A = \alpha$. It follows that the canonical form of the heat equation is given by

$$w_{\eta\eta} = \frac{1}{\alpha} w_\xi$$

1.4 Elliptic equations

For an elliptic PDE the discriminant $\Delta = B^2 - 4AC < 0$. In this case, we have seen that, to reduce this PDE to canonical form we need to choose the new variables ξ and η to produce $b = 0$ and $a = c$, or $b = 0$ and $a - c = 0$. Then, from (8) we obtain the following equations:

$$A\left(\xi_x^2 - \eta_x^2\right) + B(\xi_x\xi_y - \eta_x\eta_y) + C\left(\xi_y^2 - \eta_y^2\right) = 0 \tag{34a}$$

$$2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = 0 \tag{34b}$$

For hyperbolic and parabolic PDEs, ξ and η are satisfied by equations that are not coupled each other (see (13) and (25)). However, equations (34) are coupled since both unknowns ξ and η appear in both equations. In an attempt to separate them, we add the first of these equation to complex number i times the second to give

$$A(\xi_x + i\eta_x)^2 + B(\xi_x + i\eta_x)(\xi_y + i\eta_y) + C(\xi_y + i\eta_y)^2 = 0$$

Dividing the above equation throughout by $(\xi_y + i\eta_y)^2$ to obtain

$$A\left(\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y}\right)^2 + B\left(\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y}\right) + C = 0 \tag{35}$$

This equation can be solved for two possible values of the ratio

$$\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-B \pm i\sqrt{4AC - B^2}}{2A} \tag{36}$$

Clearly, these two roots are complex conjugates and are given by

$$\frac{\alpha_x}{\alpha_y} = \frac{-B + i\sqrt{4AC - B^2}}{2A} \tag{37a}$$

$$\frac{\beta_x}{\beta_y} = \frac{-B - i\sqrt{4AC - B^2}}{2A} \tag{37b}$$

where $\beta(x,y)$ is the complex conjugate of $\alpha(x,y)$. They are given by

$$\alpha(x,y) = \xi(x,y) + i\eta(x,y) \tag{38a}$$
$$\beta(x,y) = \xi(x,y) - i\eta(x,y) \tag{38b}$$

We will now proceed in a purely formal fashion. As the total derivative of α along the coordinate line $\alpha(x,y) =$ constant, $d\alpha = 0$. It follows that

$$d\alpha = \alpha_x dx + \alpha_y dy = 0$$

and hence, the slope of such curves is given by

$$\frac{dy}{dx} = -\frac{\alpha_x}{\alpha_y}$$

We also have a similar result along coordinate line $\beta(x,y) =$ constant, i.e.,

$$\frac{dy}{dx} = -\frac{\beta_x}{\beta_y}$$

From the foregoing discussion it follows that

$$\frac{dy}{dx} = \lambda_1 = \frac{B - i\sqrt{4AC - B^2}}{2A} \tag{39a}$$
$$\frac{dy}{dx} = \lambda_2 = \frac{B + i\sqrt{4AC - B^2}}{2A} \tag{39b}$$

Equations (39a) and (39b) are called the characteristic equation of the PDE (2). Clearly, the solution of this differential equations are necessarily complex-valued and as a consequence there are no real characteristic exist for an elliptic PDE.

The complex variables α and β are determined by the respective solutions of the two ordinary differential equations (39a) and (39b). Integration of equation (39a) leads to the family of curvilinear coordinates $\alpha(x,y) = c_1$ while the integration of (39b) gives another family of curvilinear coordinates $\beta(x,y) = c_2$, where c_1 and c_2 are complex constants of integration. Since α and β are complex function the characteristic curves of the elliptic equation (2) are not real.

Now the real and imaginary parts of α and β give the required transformation variables ξ and η. Thus, we have

$$\xi = \frac{\alpha + \beta}{2} \qquad \eta = \frac{\alpha - \beta}{2i} \tag{40}$$

With the choice of coordinate variables (40), equation (7) reduces to following canonical form:

$$w_{\xi\xi} + w_{\eta\eta} = \psi\left(\xi, \eta, w, w_\xi, w_\eta\right) \tag{41}$$

where $\psi = \phi/a$.

Note: It may be noted that the quasilinear second-order equations in two independent variables can also be classified in a similar way according to rule analogous to those developed above for semilinear equations. However, since A, B, and C are now functions of u_x, u_y, and u its type turns out to depend in general on the particular solution searched and not just on the values of the independent variables.

Example 4

Show that the equation

$$u_{xx} + x^2 u_{yy} = 0$$

is elliptic everywhere except on the coordinate axis $x = 0$, find the characteristic variables and hence write the equation in canonical form.

Solution The given equation is of the form (2) where

$$A = 1, \qquad B = 0, \qquad C = x^2$$

The discriminant, $\Delta = B^2 - 4AC = -4x^2 < 0$ for $x \neq 0$, and therefore the PDE is elliptic. The roots of the characteristic polynomial are given by

$$\lambda_1 = \frac{B - i\sqrt{4AC - B^2}}{2A} = -ix \qquad \text{and} \qquad \lambda_2 = \frac{B + i\sqrt{4AC - B^2}}{2A} = ix$$

Therefore, from the characteristic equations (18a) and (18b), we have

$$\frac{dy}{dx} = -ix, \qquad \frac{dy}{dx} = ix$$

Integrating the above two ODEs to obtain the characteristics of the wave equation

$$y = -i\frac{x^2}{2} + c_1, \qquad y = i\frac{x^2}{2} + c_2$$

where c_1 and c_2 are the complex constants. We see that the two families of complex characteristics for the elliptic equation are given by $y + ix^2/2 =$ constant and $y - ix^2/2 =$ constant. It follows, then, that the transformation

$$\alpha = y + i\frac{x^2}{2}, \qquad \beta = y - i\frac{x^2}{2}$$

The real and imaginary parts of α and β give the required transformation variables ξ and η. Thus, we have

$$\xi = \frac{\alpha + \beta}{2} = y \qquad \eta = \frac{\alpha - \beta}{2i} = \frac{x^2}{2}$$

With this choice of coordinate variables, equation (7) reduces to following canonical form. From the relations (6), we have

$$\begin{aligned} u_{xx} &= w_{\xi\xi}\xi_x^2 + 2w_{\xi\eta}\xi_x\eta_x + w_{\eta\eta}\eta_x^2 + w_\xi\xi_{xx} + w_\eta\eta_{xx} \\ &= x^2 w_{\eta\eta} + w_\eta \\ u_{yy} &= w_{\xi\xi}\xi_y^2 + 2w_{\xi\eta}\xi_y\eta_y + w_{\eta\eta}\eta_y^2 + w_\xi\xi_{yy} + w_\eta\eta_{yy} \\ &= w_{\xi\xi} \end{aligned}$$

Substituting these relations in the given PDE and noting that $x^2 = 2\eta$, we obtain

$$w_{\xi\xi} + w_{\eta\eta} = -\frac{1}{2\eta}w_\eta$$

This is the canonical form of the given hyperbolic PDE. Therefore, the PDE

$$u_{xx} + x^2 u_{yy} = 0$$

in rectangular coordinate system (x,y) has been transformed to PDE

$$w_{\xi\xi} + w_{\eta\eta} = -\frac{1}{2\eta} w_\eta$$

in curvilinear coordinate system (ξ,η). Here $\xi =$ const. lines represents a family of straight lines parallel to x axis and $\eta =$ const. lines represents family of parabolas.

Example 5

Consider the Tricomi equation

$$u_{xx} - x u_{yy} = 0$$

This is simple model of a second-order PDE of mixed elliptic-hyperbolic type with two independent variables. The Tricomi equation is a prototype of the Chaplygin's equation for study of transonic flow.

The Tricomi equation is of the form (2) where

$$A = 1, \qquad B = 0, \qquad C = -x$$

The discriminant, $\Delta = B^2 - 4AC = 4x$. Therefore, the Tricomi equation is hyperbolic for $x > 0$, elliptic for $x < 0$ and degenerates to an equation of parabolic type on the line $x = 0$. Assuming $x > 0$, the roots of the characteristic polynomial are given by

$$\lambda_1 = \frac{B + \sqrt{\Delta}}{2A} = \sqrt{x} \qquad \text{and} \qquad \lambda_2 = \frac{B - \sqrt{\Delta}}{2A} = -\sqrt{x}$$

Therefore, from the characteristic equations (18a) and (18b), we have

$$\frac{dy}{dx} = \sqrt{x}, \qquad \frac{dy}{dx} = -\sqrt{x}$$

Integrating the above two ODEs to obtain the characteristics of the wave equation

$$y = \tfrac{2}{3}x^{3/2} + c_1, \qquad y = -\tfrac{2}{3}x^{3/2} + c_2$$

where c_1 and c_2 are the constants of integration. We see that the two families of characteristics for the wave equation are given by $y - 2/(3x^{3/2}) =$ constant and $y + 2/(3x^{3/2}) =$ constant. It follows, then, that the transformation

$$\xi = y - \tfrac{2}{3}x^{3/2}, \qquad \eta = y + \tfrac{2}{3}x^{3/2}$$

reduces the wave equation to canonical form. The derivatives of ξ and η are given by

$$\begin{aligned} \xi_x &= -\sqrt{x} & \xi_y &= 1 \\ \xi_{xx} &= -\tfrac{1}{2\sqrt{x}} & \xi_{yy} &= 0 \\ \eta_x &= \sqrt{x} & \eta_y &= 1 \\ \eta_{xx} &= \tfrac{1}{2\sqrt{x}} & \eta_{yy} &= 0 \end{aligned}$$

From the relations (6), we have

$$\begin{aligned}
u_{xx} &= w_{\xi\xi}\xi_x^2 + 2w_{\xi\eta}\xi_x\eta_x + w_{\eta\eta}\eta_x^2 + w_\xi\xi_{xx} + w_\eta\eta_{xx} \\
&= xw_{\xi\xi} - 2xw_{\xi\eta} + xw_{\eta\eta} - \frac{1}{2\sqrt{x}}w_\xi + \frac{1}{2\sqrt{x}}w_\eta \\
u_{yy} &= w_{\xi\xi}\xi_y^2 + 2w_{\xi\eta}\xi_y\eta_y + w_{\eta\eta}\eta_y^2 + w_\xi\xi_{yy} + w_\eta\eta_{yy} \\
&= w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}
\end{aligned}$$

We also have the following relation

$$x^{3/2} = \frac{3(\eta-\xi)}{4}$$

Substituting these relations in the given PDE to obtain

$$w_{\xi\eta} = \frac{1}{6}\frac{w_\xi - w_\eta}{\xi-\eta}$$

This is the canonical form of the Tricomi equation in the hyperbolic region.

Example 6

An interesting example is provided by the stationary potential flow equation in two dimensions, defined by equation (where c designates the speed of sound):

$$\left(1-\frac{u^2}{c^2}\right)\frac{\partial^2\phi}{\partial x^2} - 2\frac{uv}{c^2}\frac{\partial^2\phi}{\partial x\partial y} + \left(1-\frac{v^2}{c^2}\right)\frac{\partial^2\phi}{\partial y^2} = 0$$

with

$$A = \left(1-\frac{u^2}{c^2}\right), \qquad B = -2\frac{uv}{c^2}, \qquad C = \left(1-\frac{v^2}{c^2}\right)$$

we can write the potential equation under the form (2). In this particular case the discriminant (B^2-4AC) becomes, introducing the Mach number, $M(=\sqrt{u^2+v^2}/c)$

$$B^2-4AC = 4\left(\frac{u^2+v^2}{c^2}-1\right) = 4\left(M^2-1\right)$$

and hence the stationary potential equation is elliptic for subsonic flows and hyperbolic for supersonic flows. Along the sonic line $M=1$, the equation is parabolic. This mixed nature of the potential equation has been a great challenge for the numerical computation of transonic flows since the transition line between the subsonic and the supersonic regions is part of the solution. An additional complication arises from the presence of shock waves which are discontinuities of the potential derivatives and which can arise in the supersonic regions.

2 Classification of Second-Order Equations in n Variables

We now consider the classification to second-order PDEs in more than two independent variables. To extend the examination of characteristics for more than two independent variables is less useful. In n dimension, we need to consider $(n-1)$ dimensional surfaces. In three dimensions, it is necessary to obtain transformations, $\xi = \xi(x,y,z)$, $\eta = \eta(x,y,z)$, and $\zeta = \zeta(x,y,z)$ such that all cross derivatives in (ξ, η, ζ) disappear. However, this approach will fail for more than three independent variables and hence it is not usually possible to reduce the equation to a simple canonical form. Consider a general second-order semilinear partial differential equation in n independent variables

$$\begin{array}{ccccccccccc}
a_{11}\dfrac{\partial^2 u}{\partial x_1 \partial x_1} & + & a_{12}\dfrac{\partial^2 u}{\partial x_1 \partial x_2} & + & a_{13}\dfrac{\partial^2 u}{\partial x_1 \partial x_3} & + & \cdots & + & a_{1n}\dfrac{\partial^2 u}{\partial x_1 \partial x_n} & + & \\
a_{21}\dfrac{\partial^2 u}{\partial x_2 \partial x_1} & + & a_{22}\dfrac{\partial^2 u}{\partial x_2 \partial x_2} & + & a_{23}\dfrac{\partial^2 u}{\partial x_2 \partial x_3} & + & \cdots & + & a_{2n}\dfrac{\partial^2 u}{\partial x_2 \partial x_n} & + & \\
\vdots & & \vdots & & \vdots & & \ddots & & \vdots & & \\
a_{n1}\dfrac{\partial^2 u}{\partial x_n \partial x_1} & + & a_{n2}\dfrac{\partial^2 u}{\partial x_n \partial x_2} & + & a_{n3}\dfrac{\partial^2 u}{\partial x_n \partial x_3} & + & \cdots & + & a_{nn}\dfrac{\partial^2 u}{\partial x_n \partial x_n} & + & \\
b_1\dfrac{\partial u}{\partial x_1} & + & b_2\dfrac{\partial u}{\partial x_2} & + & b_3\dfrac{\partial u}{\partial x_3} & + & \cdots & + & b_n\dfrac{\partial u}{\partial x_n} & + & cu + d = 0
\end{array}$$

For more than three independent variables it is convenient to write the above PDE in the following form:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu + d = 0 \tag{42}$$

where the coefficients a_{ij}, b_i, c, d are functions of $x = (x_1, x_2, \ldots, x_n)$, $u = u(x_1, x_2, \ldots, x_n)$, and n is the number of independent variables. Equation (42) can be written in matrix form as

$$\begin{bmatrix} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix} + \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix} + cu + d = 0$$

We assume that the coefficient matrix $A = (a_{ij})$ to be symmetric. If A is not symmetric, we can always find a symmetric matrix $\overline{a}_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$ such that (42) can be rewritten as

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{a}_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu + d = 0$$

For example, consider the equation

$$\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2} = f(x_1, x_2)$$

Since

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = \frac{\partial^2 u}{\partial x_2 \partial x_1}$$

we may write

$$\frac{\partial^2 u}{\partial x_1^2} - \frac{1}{2}\frac{\partial^2 u}{\partial x_1 \partial x_2} - \frac{1}{2}\frac{\partial^2 u}{\partial x_2 \partial x_1} + \frac{\partial^2 u}{\partial x_2^2} = f(x_1, x_2)$$

or in matrix form

$$\begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \end{bmatrix} \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \end{bmatrix} = f(x_1, x_2)$$

Comparing to the general equation in matrix form above, we can see that the coefficient matrix A is now symmetric.

Now consider the transformation

$$\xi = Qx$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $Q = (q_{ij})$ is an $n \times n$ arbitrary matrix. Using index notation, this transformation can be written as

$$\xi_i = \sum_{j=1}^{n} q_{ij} x_j$$

Repeated application of chain rule in the forms

$$\frac{\partial}{\partial x_i} = \sum_{k=1}^{n} \frac{\partial}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_i}$$

and

$$\frac{\partial^2}{\partial x_i \partial x_j} = \sum_{k,l=1}^{n} \frac{\partial^2}{\partial \xi_k \partial \xi_l} \frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_l}{\partial x_j}$$

to the derivatives of $u(x_1, x_2, \ldots, x_n)$ in (42) with respect to $x_1, x_2, \ldots, x_n$ transforms them into derivatives of $w(\xi_1, \xi_2, \ldots, \xi_n)$ with respect to $\xi_1, \xi_2, \ldots, \xi_n$. This allows equation (42) to be expressed as

$$\sum_{k,l=1}^{n} \left(\sum_{i,j=1}^{n} q_{ki} a_{ij} q_{lj} \right) \frac{\partial^2 u}{\partial \xi_k \partial \xi_l} + \text{lower-order terms} = 0 \tag{43}$$

The coefficient matrix of the terms $\partial^2 u / (\partial \xi_k \partial \xi_l)$ in this transformed expression is seen to be equal to $Q^T A Q$. That is,

$$(q_{ki} a_{ij} q_{lj}) \equiv Q^T A Q$$

From linear algebra, we know that for any real symmetric matrix A, there is an associate orthogonal matrix P such that $P^T A P = \Lambda$. Here P is called diagonalizing matrix of A and Λ is a diagonal matrix whose element are the eigenvalues, λ_i, of A and the columns of P the linearly independent eigenvectors of A, $e_i = (e_{1i}, e_{2i}, \ldots, e_{ni})$. So, we have

$$P = (e_{ij}) \qquad \text{and} \qquad \Lambda = (\lambda_i \delta_{ij}), \quad i, j = 1, 2, \ldots, n$$

where δ_{ij} is the Kronecker delta. Now if the transformation is such that Q is taken to be a diagonalizing matrix of A, it follows that

$$Q^T A Q = \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \tag{44}$$

The numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real, because the eigenvalues of a real symmetric matrix are always real. It is instructive to note that the previously mentioned transformation (for second-order PDE with two independent variables) to remove cross derivatives is equivalent to finding eigenvalues λ_i of the coefficient matrix A.

We are now in a position to classify the equation (42).

- Equation is called elliptic if all eigenvalues λ_i of A are non-zero and have the same sign.
- Equation is called hyperbolic if all eigenvalues λ_i of A are non-zero and have the same sign except for one of the eigenvalues.
- Equation is called parabolic if any of the eigenvalues λ_i of A is zero. This means that the coefficient matrix A is singular.

When more than two independent variables are involved, there are other intermediate classifications exist which depends on the number of zero eigenvalues and the pattern of signs of the non-zero eigenvalues. These sub classification has not much practical importance and will not be discussed here.

Example 7

Classify the three-dimensional Laplace equation

$$u_{xx} + u_{yy} + u_{zz} = 0$$

Solution The coefficient matrix is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As the coefficient matrix is already in diagonalized form it can be seen immediately that it has three non-zero eigenvalues which are all positive. Hence, according to the classification rule the given PDE is elliptic.

Example 8

Classify the two-dimensional wave equation

$$u_{tt} - c^2 (u_{xx} + u_{yy}) = 0$$

Solution The coefficient matrix is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -c^2 & 0 \\ 0 & 0 & -c^2 \end{bmatrix}$$

As the coefficient matrix is already in diagonalized form it can be seen immediately that it has three non-zero eigenvalues which are all negative except one. Hence, according to the classification rule the given PDE is hyperbolic.

Example 9

Classify the two-dimensional heat equation

$$u_t - \alpha\,(u_{xx} + u_{yy}) = 0$$

Solution The coefficient matrix is given by

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & -\alpha \end{bmatrix}$$

As the coefficient matrix is already in diagonalized form it can be seen immediately that it has a zero eigenvalue. Hence, according to the classification rule the given PDE is parabolic.

Example 10

Classify the two-dimensional equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

Solution First of we write the given equation in the following form:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}\frac{\partial^2 u}{\partial x \partial y} - \frac{1}{2}\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

The coefficient matrix is then given by

$$A = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

Since the coefficient matrix not in diagonalized form, we solve the eigenvalue problem $\det(A - \lambda I) = 0$. That is,

$$\begin{vmatrix} 1-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & 1-\lambda \end{vmatrix} = 0$$

Expanding the determinant to obtain

$$(1-\lambda)^2 - \frac{1}{4} = 0 \qquad \Longrightarrow \qquad \lambda^2 - 2\lambda + \frac{3}{4} = 0$$

Hence the two eigenvalues are $\lambda_1 = 1/2$ and $\lambda_2 = 3/2$. The equation is elliptic.

Example 11

Classify the following small disturbance potential equation for compressible flows:

$$\left(1-M_\infty^2\right)\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

Solution The coefficient matrix is then given by

$$A=\begin{bmatrix} 1-M_\infty^2 & 0 \\ 0 & 1 \end{bmatrix}$$

As the coefficient matrix is already in diagonalized form it can be seen immediately that its eigenvalues are given by

$$\lambda_1=1, \qquad \lambda_2=1-M_\infty^2$$

It follows that: If $M_\infty^2<1$ then all eigenvalues are nonzero and of same sign thus small disturbance potential equation is elliptic, if $M_\infty^2=1$ then one of the eigenvalues is zero thus small disturbance potential equation is parabolic, and if $M_\infty^2>1$ then all eigenvalues are nonzero and are of opposite sign thus small disturbance potential equation is hyperbolic.

3 Classification of First-Order System of Equations

Consider the semilinear[1] first-order system of two equations in two dependent variables (u,v) and three independent variables (x,y,z) (corresponding to a thee-dimensional space) as given below:

$$\begin{aligned} a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial v}{\partial x}+b_{11}\frac{\partial u}{\partial y}+b_{12}\frac{\partial v}{\partial y}+c_{11}\frac{\partial u}{\partial z}+c_{12}\frac{\partial v}{\partial z} &= f_1 \\ a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial v}{\partial x}+b_{21}\frac{\partial u}{\partial y}+b_{22}\frac{\partial v}{\partial y}+c_{21}\frac{\partial u}{\partial z}+c_{22}\frac{\partial v}{\partial z} &= f_2 \end{aligned}$$

In matrix form, we have

$$\begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} & c_{11} & c_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} & c_{21} & c_{22} \end{bmatrix}\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial z} \end{bmatrix}=\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

Alternatively, the above system can be rewritten as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} u \\ v \end{bmatrix}+\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\frac{\partial}{\partial y}\begin{bmatrix} u \\ v \end{bmatrix}+\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\frac{\partial}{\partial z}\begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

[1]For a first-order PDE to be semilinear, $a_{ij}=a_{ij}(x,y,z)$, $b_{ij}=b_{ij}(x,y,z)$, $c_{ij}=c_{ij}(x,y,z)$ and $f_i=f_i(x,y,z,u,v)$.

or

$$A(x,y,z)\frac{\partial U}{\partial x} + B(x,y,z)\frac{\partial U}{\partial y} + C(x,y,z)\frac{\partial U}{\partial z} = F(x,y,z,U)$$

where

$$U = \begin{bmatrix} u \\ v \end{bmatrix} \qquad F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

Using indicial notations, the system of first-order PDE for two dependent variables and three independent variables can be written as

$$\sum_{j=1}^{2}\left(a_{ij}\frac{\partial u_j}{\partial x} + b_{ij}\frac{\partial u_j}{\partial y} + c_{ij}\frac{\partial u_j}{\partial z}\right) = f_i, \qquad i = 1,2$$

The system of first-order PDE of three independent variables can be generalized for n dependent variables, u_j, as follows:

$$\sum_{j=1}^{n}\left(a_{ij}\frac{\partial u_j}{\partial x} + b_{ij}\frac{\partial u_j}{\partial y} + c_{ij}\frac{\partial u_j}{\partial z}\right) = f_i, \qquad i = 1,2,\ldots,n \tag{45a}$$

or in matrix form

$$A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} + C\frac{\partial U}{\partial z} = F \tag{45b}$$

where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} \qquad C = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$$

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

Finally, we consider the most general form of system of first-order PDEs. Suppose we have n dependent variables u_j, in an m-dimensional space x_k (i.e., m independent variables), we can group all the variables u_j in an $(n \times 1)$ column vector U and write the system of first-order PDEs

$$\sum_{k=1}^{m} A^k \frac{\partial U}{\partial x_k} = F \tag{46a}$$

or

$$\sum_{k=1}^{m}\sum_{j=1}^{n} a_{ij}^k \frac{\partial u_j}{\partial x_k} = f_i, \qquad i = 1,2,\ldots,n \tag{46b}$$

As an example let us consider the system n equations (and an equal number of dependent variables) in two dimensions (x,y):

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} \frac{\partial}{\partial y} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

or

$$A(x,y)\frac{\partial U}{\partial x} + B(x,y)\frac{\partial U}{\partial y} = F(x,y,U) \tag{47}$$

Assuming that A is nonsingular, the system (47) can be written in a more convenient form by pre-multiplying by A^{-1}:

$$\frac{\partial U}{\partial x} + D(x,y)\frac{\partial U}{\partial y} = E(x,y,U) \tag{48}$$

or

$$\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & \ddots & \vdots \\ d_{n1} & \cdots & d_{nn} \end{bmatrix} \frac{\partial}{\partial y} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$$

where

$$D = A^{-1}B \qquad \text{and} \qquad E = A^{-1}F$$

Using indicial notation, the system (48) can be written as

$$\frac{\partial u_j}{\partial x} + \sum_{j=1}^{n} d_{ij}\frac{\partial u_j}{\partial y} = e_i, \qquad i = 1,2,\ldots,n$$

and in component form, the system becomes

$$\begin{array}{ccccccccccc}
\dfrac{\partial u_1}{\partial x} & + & d_{11}\dfrac{\partial u_1}{\partial y} & + & d_{12}\dfrac{\partial u_2}{\partial y} & + & \cdots & + & d_{1n}\dfrac{\partial u_n}{\partial y} & = & e_1 \\
\dfrac{\partial u_2}{\partial x} & + & d_{21}\dfrac{\partial u_1}{\partial y} & + & d_{22}\dfrac{\partial u_2}{\partial y} & + & \cdots & + & d_{2n}\dfrac{\partial u_n}{\partial y} & = & e_2 \\
\vdots & & \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
\dfrac{\partial u_n}{\partial x} & + & d_{n1}\dfrac{\partial u_1}{\partial y} & + & d_{n2}\dfrac{\partial u_2}{\partial y} & + & \cdots & + & d_{nn}\dfrac{\partial u_n}{\partial y} & = & e_n
\end{array}$$

Just as in the case of a single partial differential equation, the important properties of solutions of the system (48) depend only on its principal part $U_x + DU_y$. Since this principal part is completely determined by the coefficient matrix $D(x,y) = A^{-1}B$, this matrix plays a fundamental role in the study of (47).

3.1 Canonical form and classification

Let us consider the transformation

$$V = P^{-1}U \qquad \Rightarrow \qquad U = PV$$

where $P = (p_{ij})$ is an $n \times n$ arbitrary nonsingular matrix. Then

$$\frac{\partial U}{\partial x} = P\frac{\partial V}{\partial x} + \frac{\partial P}{\partial x}V, \qquad \frac{\partial U}{\partial y} = P\frac{\partial V}{\partial y} + \frac{\partial P}{\partial y}V$$

Substituting these into (48) to obtain

$$P\frac{\partial V}{\partial x}+\frac{\partial P}{\partial x}V+DP\frac{\partial V}{\partial y}+D\frac{\partial P}{\partial y}V=E$$

Rearranging,

$$P\frac{\partial V}{\partial x}+DP\frac{\partial V}{\partial y}=E-\left(\frac{\partial P}{\partial x}+D\frac{\partial P}{\partial y}\right)V=G$$

We multiply the above equation by P^{-1} to obtain

$$\frac{\partial V}{\partial x}+P^{-1}DP\frac{\partial V}{\partial y}=H \tag{49}$$

where

$$H=P^{-1}G=P^{-1}\left[E-\left(\frac{\partial P}{\partial x}+D\frac{\partial P}{\partial y}\right)V\right].$$

Let Λ be the $n\times n$ diagonal matrix with diagonal entries the eigenvalues of D, and P is taken to be the $n\times n$ diagonalizing matrix of D with columns the corresponding eigenvectors of D, i.e.,

$$\Lambda=(\lambda_i\delta_{ij})=\begin{bmatrix}\lambda_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_n\end{bmatrix}, \qquad P=(p_{ij})=\begin{bmatrix}p_{11} & \cdots & p_{1n}\\ \vdots & \ddots & \vdots\\ p_{n1} & \cdots & p_{nn}\end{bmatrix}$$

where δ_{ij} is the Kronecker delta. It follows that

$$P^{-1}DP=\Lambda. \tag{50}$$

It is instructive to note that the previously mentioned transformations (for second-order PDE with two independent variables) to remove cross derivatives is equivalent to diagonalizing the matrix $D=A^{-1}B$ and then finding eigenvalues λ_i of the matrix D of the system of PDEs (47). Thus, we can write the system of equations (49) [and consequently (47)] in the following canonical form:

$$\frac{\partial V}{\partial x}+\Lambda(x,y)\frac{\partial V}{\partial y}=H(x,y,V) \tag{51a}$$

The simplicity of the canonical form (51a) becomes apparent if we write it in component form,

$$\frac{\partial v_i}{\partial x}+\lambda_i(x,y)\frac{\partial v_i}{\partial y}=h_i(x,y,v_1,\ldots,v_n), \qquad i=1,2,\ldots,n \tag{51b}$$

or

$$\begin{aligned}\frac{\partial v_1}{\partial x}+\lambda_1\frac{\partial v_1}{\partial y}&=h_1\\ \frac{\partial v_2}{\partial x}+\lambda_2\frac{\partial v_2}{\partial y}&=h_2\\ &\vdots\\ \frac{\partial v_n}{\partial x}+\lambda_n\frac{\partial v_n}{\partial y}&=h_n\end{aligned} \tag{51c}$$

It is clear that the principal part of the ith equation involves only the ith unknown v_i.

The classification of the system of first-order PDEs (47) is done based on the nature of the eigenvalues λ_i of the matrix $P^{-1}DP$, which are exactly the eigenvalues values of $D = A^{-1}B$. Recall that an eigenvalue of D is a root λ of the characteristic equation

$$|D - \lambda I| = 0.$$

The system (47) based on the nature of its eigenvalues is classified as follows:

- If all the n eigenvalues of D are real and distinct the system is called hyperbolic type.
- If all the n eigenvalues of D are complex the system is called elliptic type.
- If some of the n eigenvalues are real and other complex the system is considered as hybrid of elliptic-hyperbolic type.
- If the rank of matrix D is less than n, i.e., there are less than n real eigenvalues (some of the eigenvalues are repeated) then the system is said to be parabolic type.

For a system of PDE having only two dependent variables, we can determine the eigenvalues of matrix D analytically and state the conditions classification in an explicit manner as follows. Consider the system of two equations (with dependent variables u_1 and u_2) in two dimensions (x,y):

$$a_{11}\frac{\partial u_1}{\partial x} + a_{12}\frac{\partial u_2}{\partial x} + b_{11}\frac{\partial u_1}{\partial y} + b_{12}\frac{\partial u_2}{\partial y} = f_1 \tag{52a}$$

$$a_{21}\frac{\partial u_1}{\partial x} + a_{22}\frac{\partial u_2}{\partial x} + b_{21}\frac{\partial u_1}{\partial y} + b_{22}\frac{\partial u_2}{\partial y} = f_2 \tag{52b}$$

In matrix form,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \frac{\partial}{\partial y} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

or

$$A\frac{\partial}{\partial x} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + B\frac{\partial}{\partial y} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = F$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \qquad F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

The inverse of A is

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

where $|A|$ is the determinant of matrix A. We now compute the matrix D as

$$D = A^{-1}B = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} a_{22}b_{11} - a_{12}b_{21} & a_{22}b_{12} - a_{12}b_{22} \\ a_{11}b_{21} - a_{21}b_{11} & a_{11}b_{22} - a_{21}b_{12} \end{bmatrix}$$

so that the system (52) may be written as

$$\frac{\partial}{\partial x}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \frac{1}{|A|}\begin{bmatrix} a_{22}b_{11}-a_{12}b_{21} & a_{22}b_{12}-a_{12}b_{22} \\ a_{11}b_{21}-a_{21}b_{11} & a_{11}b_{22}-a_{21}b_{12} \end{bmatrix}\frac{\partial}{\partial y}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

To determine eigenvalues of D, we solve the following eigenvalue problem:

$$|D - \lambda I| = 0$$

Expanding the above determinant to obtain the characteristic equation

$$|A|\lambda^2 - |b|\lambda + |B| = 0 \tag{53}$$

where the determinants $|A|$, $|B|$, and $|b|$ are given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$|B| = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = b_{11}b_{22} - b_{12}b_{21}$$

$$|b| = \begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix} = a_{11}b_{22} - a_{21}b_{12} + a_{22}b_{11} - a_{12}b_{21}$$

The two roots of the quadratic equation for λ are given by

$$\lambda_{1,2} = \frac{|b| \pm \sqrt{|b|^2 - 4|A||B|}}{2|A|} \tag{54}$$

Notice that this expression has the same form as equation (17) except that a, b, and c have now become determinants. Clearly, the nature of the eigenvalues depends on the sign of discriminant $|b|^2 - 4|A||B|$. The different possibilities are given below:

- If $|b|^2 - 4|A||B| > 0$, there exists two real and distinct eigenvalues and thus the system is hyperbolic.
- If $|b|^2 - 4|A||B| < 0$, there exists two complex eigenvalues and thus the system is elliptic.
- If $|b|^2 - 4|A||B| = 0$, there is only one real eigenvalue and thus the system is parabolic.

We mention here that classification of second-order system of equations in general is very complex. It is difficult to determine the mathematical character of these systems except for simple cases.

3.1.1 A special case

When A is an identity matrix, the system of equation (47) takes the form

$$\frac{\partial U}{\partial x} + B(x,y)\frac{\partial U}{\partial y} = F(x,y,U) \tag{55}$$

so that, we have

$$D = B \qquad \text{and} \qquad E = F$$

For a system of PDE having only two dependent variables, (55) becomes

$$\frac{\partial u_1}{\partial x} + b_{11}\frac{\partial u_1}{\partial y} + b_{12}\frac{\partial u_2}{\partial y} = f_1 \tag{56a}$$

$$\frac{\partial u_2}{\partial x} + b_{21}\frac{\partial u_1}{\partial y} + b_{22}\frac{\partial u_2}{\partial y} = f_2 \tag{56b}$$

or, in matrix form

$$\frac{\partial}{\partial x}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\frac{\partial}{\partial y}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = F$$

To determine eigenvalues of D, we solve the eigenvalue problem $|B - \lambda I| = 0$. The corresponding characteristic equation is

$$\lambda^2 - |b|\lambda + |B| = 0$$

where the determinants $|B|$ and $|b|$ are given by

$$|B| = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = b_{11}b_{22} - b_{12}b_{21}$$

$$|b| = \begin{vmatrix} b_{11} & 0 \\ b_{21} & 1 \end{vmatrix} + \begin{vmatrix} 1 & b_{12} \\ 0 & b_{22} \end{vmatrix} = b_{11} + b_{22}$$

The two roots of the quadratic equation for λ are given by

$$\lambda_{1,2} = \frac{|b| \pm \sqrt{|b|^2 - 4|B|}}{2} \tag{57}$$

Example 12

Classify the single first-order equation

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = f$$

where a and b are real constants.

Solution In the standard matrix form the above equation may be written as

$$A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = F$$

where

$$A = [\ a\ ], \qquad B = [\ b\ ], \qquad U = [\ u\ ], \qquad F = [\ f\ ]$$

The D matrix, in this case, can be easily found:

$$D = A^{-1}B = [\ a^{-1}\ ][\ b\ ] = [\ b/a\ ]$$

The matrix D has the eigenvalue, $\lambda = b/a$. This is always real and hence, a single first-order PDE is always hyperbolic in the space (x, y). Note that here we have only a single eigenvalue and thus a characteristic direction.

Example 13

Classify the following system of first-order equation:

$$
\begin{aligned}
a\frac{\partial \phi}{\partial x} + c\frac{\partial \psi}{\partial y} &= f_1 \\
b\frac{\partial \psi}{\partial x} + d\frac{\partial \phi}{\partial y} &= f_2
\end{aligned}
$$

The matrix form of the system is:

$$
\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \phi \\ \psi \end{bmatrix} + \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} \frac{\partial}{\partial y} \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}
$$

This equation may be written as

$$
A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = F
$$

where

$$
A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}, \qquad U = \begin{bmatrix} \phi \\ \psi \end{bmatrix}, \qquad F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}
$$

Therefore, we have

$$
\begin{array}{llll}
a_{11} = a & a_{12} = 0 & a_{21} = 0 & a_{22} = b \\
b_{11} = 0 & b_{12} = c & b_{21} = d & b_{22} = 0
\end{array}
$$

The relevant determinants can be now evaluated as

$$
\begin{aligned}
|A| &= a_{11}a_{22} - a_{12}a_{21} = ab \\
|B| &= b_{11}b_{22} - b_{12}b_{21} = -cd \\
|b| &= a_{11}b_{22} - a_{21}b_{12} + a_{22}b_{11} - a_{12}b_{21} = 0
\end{aligned}
$$

and the D matrix is given by

$$
D = A^{-1}B = \frac{1}{|A|}\begin{bmatrix} a_{22}b_{11} - a_{12}b_{21} & a_{22}b_{12} - a_{12}b_{22} \\ a_{11}b_{21} - a_{21}b_{11} & a_{11}b_{22} - a_{21}b_{12} \end{bmatrix} = \frac{1}{ab}\begin{bmatrix} 0 & bc \\ ad & 0 \end{bmatrix} = \begin{bmatrix} 0 & c/a \\ d/b & 0 \end{bmatrix}
$$

so that the system (52) may be written as

$$
\frac{\partial}{\partial x}\begin{bmatrix} \phi \\ \psi \end{bmatrix} + \begin{bmatrix} 0 & c/a \\ d/b & 0 \end{bmatrix} \frac{\partial}{\partial y}\begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}
$$

The eigenvalues of $D = A^{-1}B$ are given by

$$
\lambda_{1,2} = \frac{|b| \pm \sqrt{|b|^2 - 4|A||B|}}{2|A|} = \pm\sqrt{\frac{cd}{ab}}
$$

If $cd/ab > 0$ then the eigenvalues are real and distinct and the system is hyperbolic in the space (x,y). For instance, $a = b = 1$; $c = d = 1$ with vanishing right-hand side, the system of equation becomes

$$\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} = 0$$
$$\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} = 0$$

By eliminating the variable ψ and replacing y by t, we obtain the well-known wave equation in ϕ:

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2}$$

which is a hyperbolic equation as seen previously.

If $cd/ab < 0$ then the eigenvalues are complex and the system is elliptic in the space (x,y). For instance, $a = b = 1$; $c = -d = -1$ and vanishing right-hand side, the system of equation becomes the well-known *Cauchy–Riemann equation*:

$$\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y} = 0$$
$$\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} = 0$$

By eliminating the variable ψ, we obtain the Laplace equation in ϕ:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

which is the standard form of elliptic equations and describes steady-state diffusion phenomena. Note that we could also obtain the Laplace equation in ψ by eliminating the variable ϕ.

Finally, if one of the coefficients is equal zero, say c, then there is only one real eigenvalue and the system is parabolic. For instance, with $a = -b = 1$, $c = 0$, $d = 1$ and $f_1 = \psi$, $f_2 = 0$, the system of equation becomes

$$\frac{\partial \phi}{\partial x} = \psi$$
$$\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} = 0$$

which on eliminating the variable ϕ and replacing y by t leads to the standard form for a parabolic equation:

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$$

This is recognizable by the fact that the equation presents a combination of first and second-order derivatives.

Example 14

Let us find out the canonical form the system of first-order equations

$$\begin{aligned} \frac{\partial u_1}{\partial x} + c\frac{\partial u_2}{\partial y} &= 0 \\ \frac{\partial u_2}{\partial x} + d\frac{\partial u_1}{\partial y} &= 0 \end{aligned} \tag{58}$$

Solution The given system is of the form (55) and can be written as

$$\frac{\partial}{\partial x}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}\frac{\partial}{\partial y}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = F$$

where,

$$B = \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}, \qquad U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad F = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, we have

$$b_{11} = 0 \qquad b_{12} = c \qquad b_{21} = d \qquad b_{22} = 0$$

The relevant determinants can be now evaluated:

$$\begin{aligned} |B| &= b_{11}b_{22} - b_{12}b_{21} = -cd \\ |b| &= b_{11} + b_{22} = 0 \end{aligned}$$

The eigenvalues of B are given by

$$\lambda_{1,2} = \frac{|b| \pm \sqrt{|b|^2 - 4|B|}}{2} = \pm\sqrt{cd}$$

The eigenvector for $\lambda_1 = \sqrt{cd}$ can be found as follows

$$\begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}\begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix} = \sqrt{cd}\begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix} = \begin{bmatrix} \sqrt{c} \\ \sqrt{d} \end{bmatrix}$$

Similarly, the eigenvector for $\lambda_2 = -\sqrt{cd}$ is

$$\begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}\begin{bmatrix} p_{21} \\ p_{22} \end{bmatrix} = -\sqrt{cd}\begin{bmatrix} p_{21} \\ p_{22} \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} p_{21} \\ p_{22} \end{bmatrix} = \begin{bmatrix} \sqrt{c} \\ -\sqrt{d} \end{bmatrix}$$

The P matrix can now be constructed using eigenvectors of B, that is,

$$P = \begin{bmatrix} \sqrt{c} & \sqrt{c} \\ \sqrt{d} & -\sqrt{d} \end{bmatrix}$$

The inverse of P is

$$P^{-1} = \frac{1}{|P|}\begin{bmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{bmatrix} = \frac{-1}{2\sqrt{cd}}\begin{bmatrix} -\sqrt{d} & -\sqrt{c} \\ -\sqrt{d} & \sqrt{c} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1/\sqrt{c} & 1/\sqrt{d} \\ 1/\sqrt{c} & -1/\sqrt{d} \end{bmatrix}$$

Therefore,

$$P^{-1}BP = \Lambda = \begin{bmatrix} \sqrt{cd} & 0 \\ 0 & -\sqrt{cd} \end{bmatrix}$$

The canonical form is then given by

$$\frac{\partial V}{\partial x} + \Lambda \frac{\partial V}{\partial y} = 0$$

or

$$\frac{\partial}{\partial x}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} \sqrt{cd} & 0 \\ 0 & -\sqrt{cd} \end{bmatrix} \frac{\partial}{\partial y}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or in component form

$$\begin{aligned} \frac{\partial v_1}{\partial x} + \sqrt{cd}\frac{\partial v_1}{\partial y} &= 0 \\ \frac{\partial v_2}{\partial x} - \sqrt{cd}\frac{\partial v_2}{\partial y} &= 0 \end{aligned} \tag{59}$$

The canonical form (59) is particularly simple. Each equation involves only one unknown and can be easily solved by the methods of characteristics. The general solution of the system (59) is

$$\begin{aligned} v_1 &= f(x - \sqrt{cd}y) \\ v_2 &= g(x + \sqrt{cd}y) \end{aligned} \tag{60}$$

where f and g are arbitrary functions of a single variable.

The vectors U and V are related through the following transformation:

$$U = PV \qquad \Rightarrow \qquad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \sqrt{c} & \sqrt{c} \\ \sqrt{d} & -\sqrt{d} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \sqrt{c}(v_1 + v_2) \\ \sqrt{d}(v_1 - v_2) \end{bmatrix} \tag{61}$$

Now the general solution of (58) can be obtained using the relation (61) as

$$\begin{aligned} u_1 &= \sqrt{c}[f(x - \sqrt{cd}y) + g(x + \sqrt{cd}y)] \\ u_2 &= \sqrt{d}[f(x - \sqrt{cd}y) - g(x + \sqrt{cd}y)] \end{aligned} \tag{62}$$

Example 15

The one-dimensional form of the time-dependent shallow water equations can be written as

$$\begin{aligned} \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} &= 0 \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} &= 0 \end{aligned}$$

where h represents the water height, g is the gravity acceleration and u the horizontal velocity.

Solution Since A is a unit matrix, the given system in matrix form

$$\frac{\partial}{\partial t}\begin{bmatrix} h \\ u \end{bmatrix} + \begin{bmatrix} u & h \\ g & u \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} h \\ u \end{bmatrix} = 0$$

Introducing the vector

$$U = \begin{bmatrix} h \\ u \end{bmatrix}$$

the system is written in the condensed form:

$$\frac{\partial U}{\partial t} + B\frac{\partial U}{\partial x} = 0$$

where,

$$B = \begin{bmatrix} u & h \\ g & u \end{bmatrix}, \qquad U = \begin{bmatrix} h \\ u \end{bmatrix}, \qquad F = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, we have

$$b_{11} = u \qquad b_{12} = h \qquad b_{21} = g \qquad b_{22} = u$$

The relevant determinants can be now evaluated:

$$|B| = b_{11}b_{22} - b_{12}b_{21} = u^2 - gh$$
$$|b| = a_{11}b_{22} - a_{21}b_{12} + a_{22}b_{11} - a_{12}b_{21} = u + u = 2u$$

The eigenvalues of B are given by

$$\lambda_{1,2} = \frac{|b| \pm \sqrt{|b|^2 - 4|B|}}{2} = \frac{2u \pm \sqrt{4u^2 - 4(u^2 - gh)}}{2} = u \pm \sqrt{gh}$$

The eigenvector for $\lambda_1 = u + \sqrt{gh}$ can be found as follows

$$\begin{bmatrix} u & h \\ g & u \end{bmatrix}\begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix} = (u + \sqrt{gh})\begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix} = \begin{bmatrix} \sqrt{h} \\ \sqrt{g} \end{bmatrix}$$

Similarly, the eigenvector for $\lambda_2 = u - \sqrt{gh}$ is

$$\begin{bmatrix} u & h \\ g & u \end{bmatrix}\begin{bmatrix} e_{21} \\ e_{22} \end{bmatrix} = (u - \sqrt{gh})\begin{bmatrix} e_{21} \\ e_{22} \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} e_{21} \\ e_{22} \end{bmatrix} = \begin{bmatrix} -\sqrt{h} \\ \sqrt{g} \end{bmatrix}$$

The P matrix can now be constructed using eigenvectors of B, that is,

$$P = \begin{bmatrix} \sqrt{h} & -\sqrt{h} \\ \sqrt{g} & \sqrt{g} \end{bmatrix}$$

Now, the diagonal matrix Λ by definition is $P^{-1}BP$. As we have already found the eigenvalues of B, the matrix Λ can be directly obtained as

$$\Lambda = \begin{bmatrix} u + \sqrt{gh} & 0 \\ 0 & u - \sqrt{gh} \end{bmatrix}$$

The canonical form is then given by

$$\frac{\partial V}{\partial t} + \Lambda \frac{\partial V}{\partial x} = 0$$

or

$$\frac{\partial}{\partial t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} u+\sqrt{gh} & 0 \\ 0 & u-\sqrt{gh} \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or in component form

$$\begin{aligned} \frac{\partial v_1}{\partial t} + (u+\sqrt{gh})\frac{\partial v_1}{\partial x} &= 0 \\ \frac{\partial v_2}{\partial t} + (u-\sqrt{gh})\frac{\partial v_2}{\partial x} &= 0 \end{aligned}$$

Each equation involves only one unknown and can be easily solved by the methods of characteristics. The general solution of the system is

$$\begin{aligned} v_1 &= f_1[x-(u+\sqrt{gh})t] \\ v_2 &= f_2[x-(u-\sqrt{gh})t] \end{aligned} \tag{63}$$

where f_1 and f_2 are arbitrary functions of a single variable.

The vectors U and V are related through the following transformation:

$$U = PV \qquad \Rightarrow \qquad \begin{bmatrix} h \\ u \end{bmatrix} = \begin{bmatrix} \sqrt{h} & -\sqrt{h} \\ \sqrt{g} & \sqrt{g} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \sqrt{h}(v_1-v_2) \\ \sqrt{g}(v_1+v_2) \end{bmatrix}$$

Now the general solution can be obtained using the relation (63) as

$$\begin{aligned} h &= \sqrt{h}[f_1(x-(u+\sqrt{gh})t) - f_2(x-(u-\sqrt{gh})t)] \\ u &= \sqrt{g}[f_1(x-(u+\sqrt{gh})t) + f_2(x-(u-\sqrt{gh})t)] \end{aligned} \tag{64}$$

The procedure for classification of semilinear PDEs is equally applicable for quasilinear PDEs. We illustrate this with an example from fluid dynamics.

Example 16

Classify the Euler equations for unsteady, one-dimensional flow:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} &= 0 \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{c^2}{\rho}\frac{\partial \rho}{\partial x} \end{aligned}$$

where the speed of sound c is given by the isentropic relation between pressure and density as

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

Solution In matrix form the Euler equation can be written as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \rho \\ u \end{bmatrix} + \begin{bmatrix} u & \rho \\ c^2/\rho & u \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \rho \\ u \end{bmatrix} = 0$$

Introducing the vector

$$U = \begin{bmatrix} \rho \\ u \end{bmatrix}$$

the system can be written in the condensed form:

$$A\frac{\partial U}{\partial t} + B\frac{\partial U}{\partial x} = 0$$

where,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} u & \rho \\ c^2/\rho & u \end{bmatrix}, \qquad U = \begin{bmatrix} \rho \\ u \end{bmatrix}, \qquad F = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, we have

$$\begin{array}{llll} a_{11} = 1 & a_{12} = 0 & a_{21} = 0 & a_{22} = 1 \\ b_{11} = u & b_{12} = \rho & b_{21} = c^2/\rho & b_{22} = u \end{array}$$

The relevant determinants can be now evaluated:

$$\begin{aligned} |A| &= a_{11}a_{22} - a_{12}a_{21} = 1 \\ |B| &= b_{11}b_{22} - b_{12}b_{21} = u^2 - c^2 \\ |b| &= a_{11}b_{22} - a_{21}b_{12} + a_{22}b_{11} - a_{12}b_{21} = u+u = 2u \end{aligned}$$

Since A is a unit matrix, the inverse of A is A itself. We now compute the matrix D:

$$D = A^{-1}B = B = \begin{bmatrix} u & \rho \\ c^2/\rho & u \end{bmatrix}$$

The system of Euler equation can now be written as

$$\frac{\partial U}{\partial t} + D\frac{\partial U}{\partial x} = 0$$

The eigenvalues of $D = A^{-1}B$ are given by

$$\lambda_{1,2} = \frac{|b| \pm \sqrt{|b|^2 - 4|A||B|}}{2|A|} = \frac{2u \pm \sqrt{4u^2 - 4(u^2 - c^2)}}{2} = u \pm c$$

Therefore, the two characteristics of this hyperbolic system are given by

$$\frac{dx}{dt} = u \pm c$$

The eigenvector for $\lambda_1 = u + c$ can be found as follows

$$\begin{bmatrix} u & \rho \\ c^2/\rho & u \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix} = (u+c) \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix} = \begin{bmatrix} \rho \\ c \end{bmatrix}$$

Similarly, the eigenvector for $\lambda_2 = u - c$ is

$$\begin{bmatrix} u & \rho \\ c^2/\rho & u \end{bmatrix} \begin{bmatrix} e_{21} \\ e_{22} \end{bmatrix} = (u-c) \begin{bmatrix} e_{21} \\ e_{22} \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} e_{21} \\ e_{22} \end{bmatrix} = \begin{bmatrix} -\rho \\ c \end{bmatrix}$$

The P matrix can now be constructed using eigenvectors of D, that is,

$$P = \begin{bmatrix} \rho & -\rho \\ c & c \end{bmatrix}$$

Therefore,

$$P^{-1}DP = \Lambda = \begin{bmatrix} u+c & 0 \\ 0 & u-c \end{bmatrix}$$

The canonical form is then given by

$$\frac{\partial V}{\partial t} + \Lambda \frac{\partial V}{\partial x} = 0$$

or

$$\frac{\partial}{\partial t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} u+c & 0 \\ 0 & u-c \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or in component form

$$\frac{\partial v_1}{\partial t} + (u+c)\frac{\partial v_1}{\partial x} = 0$$
$$\frac{\partial v_2}{\partial t} + (u-c)\frac{\partial v_2}{\partial x} = 0$$

Each equation involves only one unknown and can be easily solved by the standard methods of characteristics. The general solution of the system is

$$\begin{aligned} v_1 &= f_1[x - (u+c)t] \\ v_2 &= f_2[x - (u-c)t] \end{aligned} \tag{65}$$

where f_1 and f_2 are arbitrary functions of a single variable.

The vectors U and V are related through the following transformation:

$$U = PV \qquad \Rightarrow \qquad \begin{bmatrix} h \\ u \end{bmatrix} = \begin{bmatrix} \rho & -\rho \\ c & c \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \rho(v_1 - v_2) \\ c(v_1 + v_2) \end{bmatrix}$$

Now the general solution can be obtained using the relation (65) as

$$\begin{aligned} \rho &= \rho[f_1(x - (u+c)t) - f_2(x - (u-c)t)] \\ u &= c[f_1(x - (u+c)t) + f_2(x - (u-c)t)] \end{aligned} \tag{66}$$

Note: Since both the eigenvalues are real, for *all* values of the velocity u, the system is always hyperbolic in space and time. This is an extremely important property that the steady isentropic Euler equations are elliptic in the space (x, y) for subsonic velocities and hyperbolic in the space (x, y) for supersonic velocities.

Here, in space and time, the inviscid isentropic equations are always hyperbolic independently of the subsonic or supersonic state of the flow. As a consequence, the same numerical algorithms can be applied for all flow velocities. On the other hand, dealing with the steady state equations, the numerical algorithms will have to adapt to the flow regime, as the mathematical nature of the system of equations is changing when passing from subsonic to supersonic, or inversely.