Cauchy–Stokes Decomposition

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It is sometimes useful to resolve the instantaneous motion of a fluid into a translation, a dilation along three perpendicular axes, and a rigid rotation of those axes. Consider the nature of a flow field in the neighborhood of some point P which is moving with an instantaneous velocity $\overline{V}$. Let $\overline{r}(t)$ be the position vector P. The instantaneous velocity at point P is given by

$$\overline{V}_P = \overline{V}(x, y, z, t)$$

Let Q be a neighboring point of P, with a position vector $\overline{r}(t) + d\overline{r}$. The instantaneous velocity at point Q may be written as

$$\overline{V}_Q = \overline{V}_Q(x + dx, y + dy, z + dz, t)$$

Using Taylor series expansion, $\overline{V}_Q$ can be written as

$$\overline{V}_Q = \overline{V}(x, y, z, t) + \frac{\partial \overline{V}}{\partial x} dx + \frac{\partial \overline{V}}{\partial y} dy + \frac{\partial \overline{V}}{\partial z} dz + \cdots$$

The above equation is a vector equation. After neglecting the higher terms, its components are given by

$$u_Q = u + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

$$v_Q = v + \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz$$

$$w_Q = w + \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$

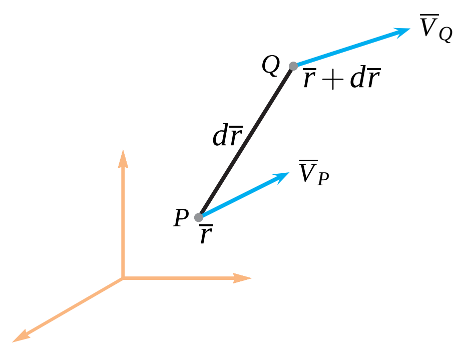

Figure 1: Velocity vectors at two neighboring points separated by a short distance.

In matrix form

$$\begin{bmatrix} u_Q \\ v_Q \\ w_Q \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

The above equation can be written in vector form as

$$\overline{V}_Q = \overline{V} + \overline{\overline{G}} \cdot d\overline{r} \tag{1}$$

where $\overline{\overline{G}} = \nabla \overline{V}$, the velocity gradient tensor and $d\overline{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$. The velocity gradient tensor can be decomposed into two parts; the symmetric tensor, $\overline{\overline{E}}$, and the skew-symmetric tensor, $\overline{\overline{R}}$. That is, $\overline{\overline{G}} = \overline{\overline{E}} + \overline{\overline{R}}$. Therefore, the equation (1) can be written as

$$\begin{aligned} \overline{V}_Q &= \overline{V} + (\overline{\overline{E}} + \overline{\overline{R}}) \cdot d\overline{r} \\ &= \overline{V} + \overline{\overline{E}} \cdot d\overline{r} + \overline{\overline{R}} \cdot d\overline{r} \end{aligned} \tag{2}$$

The symmetric tensor, $\overline{\overline{E}}$, can be identified with the strain rate tensor defined by

$$\overline{\overline{E}} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{pmatrix}$$

and the skew-symmetric tensor, $\overline{\overline{R}}$, can be identified with the rotation tensor defined by

$$\overline{\overline{R}} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & 0 & \left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) \\ \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) & \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) & 0 \end{pmatrix}$$

Using indicial notation, the i^{th} component of the equation (2) is given by

$$V_{iQ} = V_i + \varepsilon_{ij}\,dx_j + r_{ij}\,dx_j$$

Since, $r_{ij} = -\varepsilon_{ijk}\omega_k = \varepsilon_{ikj}\omega_k$, the above equation can be written as

$$V_{iQ} = V_i + \varepsilon_{ij}\,dx_j + \varepsilon_{ikj}\,\omega_k\,dx_j$$

Reverting to the vector form, we have

$$\overline{V}_Q = \overline{V} + \overline{\overline{E}} \cdot d\overline{r} + \overline{\omega} \times d\overline{r} \tag{3}$$

Finally, we write the above equation using the definition of the vorticity. Since, the vorticity, $\overline{\zeta} = 2\overline{\varpi}$, we have

$$\overline{V}_Q = \overline{V} + \overline{\overline{E}} \cdot d\overline{r} + \frac{1}{2}\overline{\zeta} \times d\overline{r} \tag{4}$$

Equation (4) represents the most general form of the movement of a fluid element. The first term $\overline{V}$ represents the translational velocity which indicates the rate of displacement of the element. The second term represents the deformation rate of the fluid element, while the third term represents the rigid body rotation of the fluid element.

Thus, *at each point in the flow the instantaneous fluid motion may be resolved into a translation, a dilation along three perpendicular axes, and a rigid rotation of those axes.* The resolution of the general motion of a fluid element into these three separate effects is called the *Cauchy–Stokes decomposition*.

Depending upon the nature of velocity gradient tensor, some special cases of fluid motion can be identified as follows.

Pure translation

For the case of pure translation, the velocity field can only be the function of time and the corresponding velocity gradient vanish. That is

$$\overline{V}(x,y,z,t) = \overline{V}_0(t)$$

and

$$\overline{\overline{G}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Pure rotation

For the case of pure rotation, the strain rates vanish and hence the velocity field is given by

$$\overline{V} = \overline{\overline{R}} \cdot \overline{r} = \frac{1}{2}\overline{\zeta} \times \overline{r} = \left[(\omega_y z - \omega_z y)\,\hat{i} + (\omega_z x - \omega_x z)\,\hat{j} + (\omega_x y - \omega_y x)\,\hat{k}\right]$$

and the velocity gradient tensor becomes

$$\overline{\overline{G}} = \overline{\overline{R}} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

In this case, $\overline{\overline{G}}$ is skew-symmetric; that is, $g_{ij} = -g_{ji}$.

Pure extensional flow

For the case of pure extensional flow, shear strain rates and rotation vanish and hence the velocity field is given by

$$\begin{aligned}\overline{V} &= \text{Diag}\left[\overline{\overline{E}}\right] \cdot \overline{r} = \varepsilon_{xx} x \hat{\imath} + \varepsilon_{yy} y \hat{\jmath} + \varepsilon_{zz} z \hat{k} \\ &= \frac{\partial u}{\partial x} x \hat{\imath} + \frac{\partial v}{\partial y} y \hat{\jmath} + \frac{\partial w}{\partial z} z \hat{k}\end{aligned}$$

and the velocity gradient tensor becomes

$$\overline{\overline{G}} = \text{Diag}\left[\overline{\overline{E}}\right] = \begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix}$$

Pure shear flow

For the case of pure shearing flow, normal strain rates and rotation vanish and hence the velocity field is given by

$$\begin{aligned}\overline{V} &= \left(\overline{\overline{E}} - \text{Diag}\left[\overline{\overline{E}}\right]\right) \cdot \overline{r} = \varepsilon_{xy} y \hat{\jmath} + \varepsilon_{xz} z \hat{k} + \varepsilon_{yx} x \hat{\imath} + \varepsilon_{yz} z \hat{k} + \varepsilon_{zx} x \hat{\imath} + \varepsilon_{zy} y \hat{\jmath} \\ &= (\varepsilon_{yx} + \varepsilon_{zx}) x \hat{\imath} + (\varepsilon_{zy} + \varepsilon_{xy}) y \hat{\jmath} + (\varepsilon_{xz} + \varepsilon_{yz}) z \hat{k}\end{aligned}$$

and the velocity gradient tensor becomes

$$\overline{\overline{G}} = \left(\overline{\overline{E}} - \text{Diag}\left[\overline{\overline{E}}\right]\right) = \begin{pmatrix} 0 & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & 0 & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & 0 \end{pmatrix}$$
