Burgers' Equation

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1 The Burgers' Equation

Burgers' equation is obtained as a result of combining nonlinear wave motion with linear diffusion and is the simplest model for analyzing combined effect of nonlinear advection and diffusion. The presence of viscous term helps suppress the wave-breaking, smooth out shock discontinuities, and hence we expect to obtain a well-behaved and smooth solution. Moreover, in the inviscid limit, as the diffusion term becomes vanishingly small, the smooth viscous solutions converge non-uniformly to the appropriate discontinuous shock wave, leading to an alternative mechanism for analyzing conservative nonlinear dynamical processes. The advection equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \tag{1}$$

propagates an initial waveform or signal at velocity c while maintaining the precise form of the initial waveform. On the other hand, the nonlinear equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{2}$$

propagates signals in such a way that distortion occurs in the waveform profile; that is, the nonlinear advection term uu_x causes either a shocking or rarefaction effect. It is well known that the heat equation

$$\frac{\partial u}{\partial t} - \alpha\frac{\partial^2 u}{\partial x^2} = 0, \qquad \alpha > 0 \tag{3}$$

contains a diffusion term αu_{xx} responsible for smoothing out the initial profile. For an initial condition of the form

$$u(x,0) = Ue^{ikx}$$

and an assumed solution of the form

$$u(x,t) = f(t)e^{ikx},$$

it is easy to obtain the solution of (3) by substitution. The solution is given by

$$u(x,t) = Ue^{-\alpha k^2 t}\sin kx. \tag{4}$$

We see here that during the time evolution of the temperature profile, it remains as a sinusoidal wave but its amplitude exponentially decreases with time as displayed in Fig. 1. Further, the waves of smaller wavelengths (or larger k) decay faster than the waves of longer wavelengths. On the other hand, for a fixed wavelength (k constant) the decay time decreases as α increases so that waves of a given wavelength attenuate faster in a medium with a larger α.

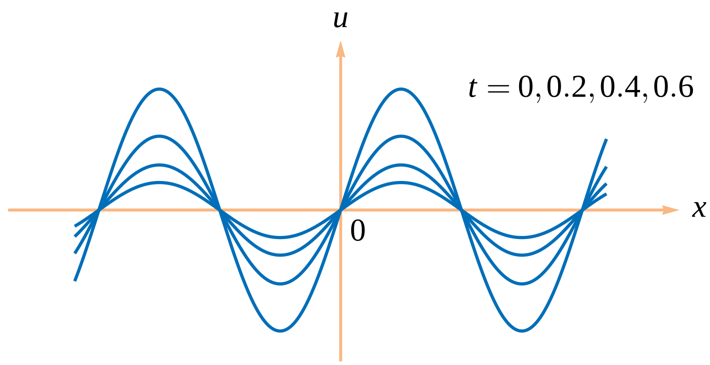

Figure 1: Transient solution (4) of heat equation.

Another useful solution of heat equation (3) in an infinite domain is obtained when the initial condition is given by a step change in u

$$u(x,0) = g(x) = \begin{cases} u_l & \text{if} \quad -\infty < x \le 0 \\ u_r & \text{if} \quad 0 < x < \infty \end{cases}$$

subject to the boundary conditions

$$\begin{aligned} u(-\infty, t) &= u_l \qquad 0 \le t < \infty \\ u(\infty, t) &= u_r \qquad 0 \le t < \infty \end{aligned}$$

Analytical solution can be easily obtained using the method of similarity transformation. It is given by

$$u(x,t) = \frac{1}{2}(u_r + u_l) + \frac{1}{2}(u_r - u_l)\,\mathrm{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right) \tag{5}$$

This shows that the presence of the diffusion term is to smooth out the initial distribution as displayed in Fig. 2.

Frequently, insight can be gained into the nature of the various terms in an evolution equation by attempting to find either traveling wave solutions

$$u(x,t) = U\cos(kx - \omega t) \tag{6a}$$

or else complex exponential solutions of the form

$$u(x,t) = Ue^{i(kx-\omega t)} \tag{6b}$$

where U is the amplitude, k the wave number, and ω the frequency.

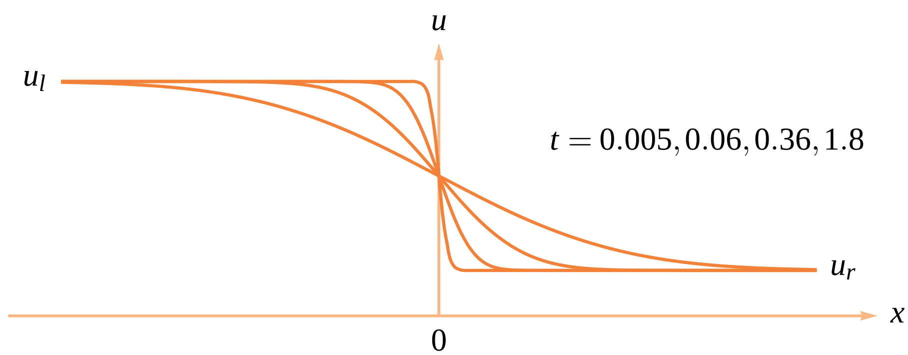

Figure 2: Transient solution (5) of heat equation.

1.1 Linear convection-diffusion equation

Consider the linear partial differential equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0, \qquad (c, \nu > 0) \tag{7}$$

which contains a linear convection term cu_x and a diffusion term νu_{xx}. For an initial condition of the form

$$u(x,0) = Ue^{ikx}$$

and an assumed solution of the form

$$u(x,t) = f(t)e^{ikx},$$

it is easy to obtain the solution of (7) by substitution. Thus, we have

$$u(x,t) = Ue^{-\nu k^2 t}e^{ik(x-ct)}. \tag{8a}$$

Since the real (or imaginary) part of the above equation is the solution, we write

$$u(x,t) = Ue^{-\nu k^2 t}\sin k(x-ct). \tag{8b}$$

The solution (8) is shown in Fig. 3. This represents a diffusive traveling wave with wavenumber k and phase velocity c. Specifically, the factor $e^{ik(k-ct)}$ represents a harmonic right traveling wave with wave number k, and the factor $Ue^{-\nu k^2 t}$ represents a decaying amplitude. As with the case of heat equation, the waves of smaller wavelengths decay faster than the waves of longer wavelengths. On the other hand, for a fixed wavelength (k constant) the decay time decreases as ν increases so that waves of a given wavelength attenuate faster in a medium with a larger ν. So, the quantity ν may be regarded as a measure of diffusion. Finally, after a sufficiently long time, only disturbances of long wavelength will survive, whereas all waves of short wavelength will decay very rapidly.

Analytical solution to (7) can also be obtained for the initial condition of the form

$$u(x,0) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0 \end{cases} \tag{9}$$

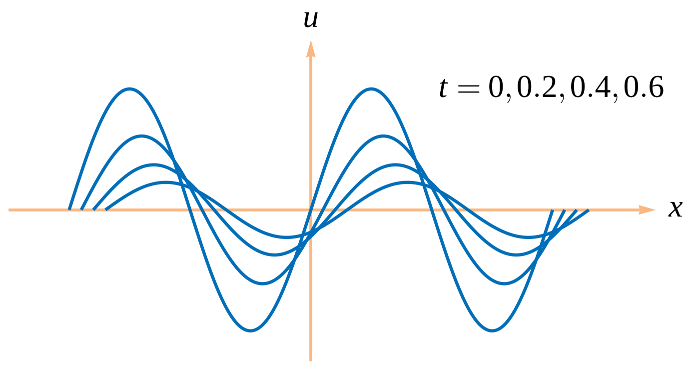

Figure 3: Traveling wave solution (8) of convection-diffusion equation.

which represents a jump discontinuity at $x = 0$. The wave front convects to the right with a speed c and its profile loses its sharpness under the influence of the diffusivity ν.

For sufficiently small values of time, the following boundary conditions may be prescribed:

$$u(-L/2, t) = 1, \qquad u(L/2, t) = 0. \tag{10}$$

Using the method separation of variables with (9) as the initial condition and (10) as the boundary conditions, an exact solution may be found:

$$u(x, t) = \frac{1}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \sin\left[(2n-1)\frac{\pi(x-ct)}{L}\right] \frac{e^{-\nu(2n-1)^2 \pi^2 t/L^2}}{2n-1}. \tag{11}$$

We note that (7) is parabolic, whereas (7) with $\nu = 0$ is hyperbolic. More importantly, the properties of the solution of the parabolic equation are significantly different from those of the hyperbolic equation.

1.2 The Burgers' equation: Travelling wave solution

Consider the nonlinear convection-diffusion equation equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0, \qquad \nu > 0 \tag{12}$$

which is known as Burgers' equation. This equation is balance between time evolution, nonlinearity, and diffusion. This is the simplest nonlinear model equation for diffusive waves in fluid dynamics. Burgers (1948) first developed this equation primarily to throw light on turbulence described by the interaction of two opposite effects of convection and diffusion.

The term uu_x will have a shocking up effect that will cause waves to break and the term νu_{xx} is a diffusion term like the one occurring in the heat equation. We attempt to find a traveling wave solution of (9) of the form

$$u(x,t) = f(\xi) = f(x-ct) \tag{13}$$

where $\xi = x - ct$ and the unknowns f and c are to be determined. By the chain rule,

$$\frac{\partial u}{\partial t} = -cf'(\xi), \qquad \frac{\partial u}{\partial x} = f'(\xi), \qquad \frac{\partial^2 u}{\partial x^2} = f''(\xi).$$

Substituting these expressions into (12) gives the ordinary differential equation

$$-cf'(\xi) + f(\xi)f'(\xi) - \nu f''(\xi) = 0. \tag{14}$$

Noting that $ff' = \frac{1}{2}df^2/ds$ and performing an integration with respect to ξ yields

$$-cf + \frac{1}{2}f^2 - \nu f' = B$$

where B is a constant of integration. Hence

$$2\nu\frac{df}{d\xi} = \left(f^2 - 2cf - 2B\right) = (f - f_1)(f - f_2) \tag{15}$$

where

$$f_1 = c + \sqrt{c^2 + 2B} \qquad \text{and} \qquad f_2 = c - \sqrt{c^2 + 2B}.$$

We assume that f_1 and f_2 are real and hence $f_1 > f_2$. Integrating (15) after separating the variables to obtain

$$\frac{\xi}{2\nu} = \int \frac{df}{(f - f_1)(f - f_2)} = \frac{1}{f_1 - f_2}\ln\frac{f_1 - f}{f - f_2}$$

where we have $f_2 < f < f_1$. This leads to the solution for $f(\xi)$ of the form

$$f(\xi) = \frac{f_1 + f_2 e^{\left(\frac{f_1 - f_2}{2\nu}\right)\xi}}{1 + e^{\left(\frac{f_1 - f_2}{2\nu}\right)\xi}} \tag{16a}$$

or

$$f(\xi) = f_2 + \frac{f_1 - f_2}{1 + e^{\left(\frac{f_1 - f_2}{2\nu}\right)\xi}}. \tag{16b}$$

Another useful expression can be written from (16a) in the form

$$\begin{aligned} f(\xi) &= \frac{1}{2}(f_1 + f_2) + \frac{f_1 + f_2 e^{\left(\frac{f_1 - f_2}{2\nu}\right)\xi}}{1 + e^{\left(\frac{f_1 - f_2}{2\nu}\right)\xi}} - \frac{1}{2}(f_1 + f_2) \\ &= \frac{1}{2}(f_1 + f_2) - \frac{1}{2}(f_1 - f_2)\tanh\left[\frac{1}{4\nu}(f_1 - f_2)\xi\right]. \end{aligned} \tag{17}$$

Thus, the explicit form of traveling wave solution to (12) is

$$u(x,t) = c - \frac{1}{2}(f_1 - f_2)\tanh\left[\frac{1}{4\nu}(f_1 - f_2)(x - ct)\right] \tag{18}$$

where the wave speed c is determined from the definition of f_1 and f_2 to be

$$c = \frac{1}{2}(f_1 + f_2). \tag{19}$$

Note that the solution (18) is independent of t in the frame of coordinates moving with $x-ct$.

Observe from (17) that

$$\lim_{\xi\to-\infty} f(\xi) = f_1, \qquad \lim_{\xi\to\infty} f(\xi) = f_2$$

and $f'(\xi) < 0$ for all ξ. Hence the solution $f(\xi)$ decreases monotonically with ξ from the constant value f_1 to the constant value f_2 as shown in Fig. 4. At $\xi = 0$, $u = \frac{1}{2}(f_1 + f_2) = c$. The wave profile $f(\xi)$ travels from left to the right, unchanged in form, with speed c equal to the average of its asymptotic values. The solution (17) of Burgers' equation is called the shock structure solution because it resembles the actual profile of a shock wave as it joins the asymptotic states f_1 and f_2. Without the viscous term the solutions of Burgers' equation would allow shocks to be formed and finally breaks. The presence of the diffusion term prevents the gradual distortion of the wave and its breaking by countering the nonlinearity. The result is a balance between the nonlinear advection term and the linear diffusion term much the same way as occurs in a real shock wave in the narrow region where the gradient is steep. The shape of the waveform (17) is significantly affected by the diffusion coefficient ν. Figure 5 shows wave profiles corresponding to $f_1 = 3.6$, $f_2 = 1.2$ for four different values of the diffusion coefficient, viz., $\nu = 1.0$, 0.4, 0.16, and 0.064. Note that the smaller ν is, the sharper the transition layer between the two asymptotic values of the solution. In the inviscid limit $\nu \to 0$, the solutions converge to the step shock wave wave solution to the inviscid Burgers'equation (2).

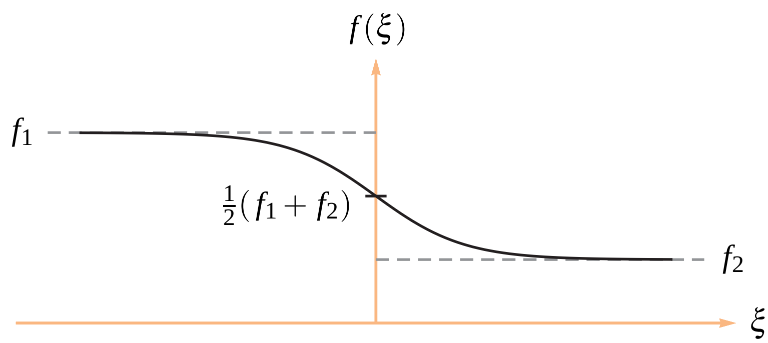

Figure 4: A typical traveling wave solution of Burgers' equation.

For the special case of $f_1 = 2$ and $f_2 = 0$, the traveling wave solution (16) (and (18)) becomes

$$f(\xi) = \frac{2}{1+e^{\frac{\xi}{\nu}}} = 1 - \tanh\left(\frac{\xi}{2\nu}\right)$$

Note that the above solution is independent of t in the frame of coordinates moving with $x-t$.

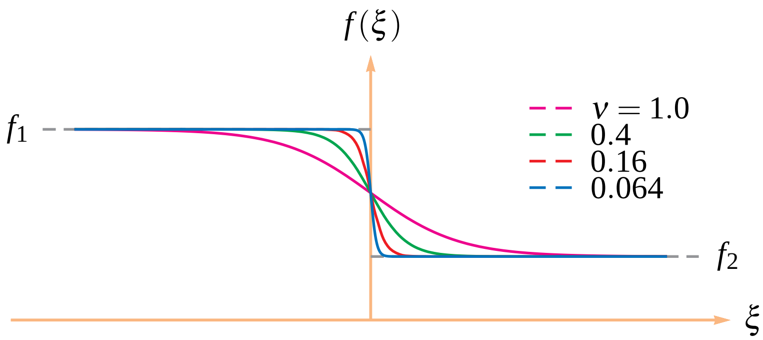

Figure 5: Traveling wave solution (17) of Burgers' equation.

1.2.1 Shock thickness

It possible to define a shock thickness δ in the traveling wave solution. To do this, we multiply both numerator and denominator of (16) by $e^{-\left(\frac{f_1-f_2}{2\nu}\right)\xi}$ to obtain

$$f(\xi) = \frac{f_2 + f_1 e^{-\left(\frac{f_1-f_2}{2\nu}\right)\xi}}{1 + e^{-\left(\frac{f_1-f_2}{2\nu}\right)\xi}}. \tag{20}$$

and for $f_1 = 2$ and $f_2 = 0$, we have

$$f(\xi) = \frac{2e^{-\frac{\xi}{\nu}}}{1 + e^{-\frac{\xi}{\nu}}}.$$

The exponential factor in this solution indicates the existence of a thin thin layer of thickness δ of the order $\nu/(f_1 - f_2)$. This thickness δ can be referred to as the shock thickness, which tends to zero as $\nu \to 0$ for fixed f_1 and f_2. Also δ increases as $f_1 \to f_2$ for a fixed ν. If δ is small compared with other typical length scales of the problem, the rapid shock transition can satisfactorily be approximated by a discontinuity. The idea of obtaining the limiting solution of (12) as $\nu \to 0$ is called the vanishing-viscosity approach to defining a sensible solution to the hyperbolic equation.

1.3 The Burgers' equation: The Cole-Hopf transformation

It is remarkable that Burgers' equation may be solved exactly using a trick discovered independently by Eberhard Hopf (1950) and Julian Cole (1951). They showed that the Burgers' equation may be transformed to the linear diffusion equation by a nonlinear transformation. This discovery, known as Cole-Hopf transformation, was a milestone in the modern era of nonlinear partial differential equations.

We consider the Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \tag{21}$$

Finding a way to transform a nonlinear differential equation into a linear equation is not an easy task. However, nonlinearizing a linear equation is quiet easy; any nonlinear transformation of variables can accomplish this task. Therefore, our starting point is the linear heat equation of the form (3)

$$\frac{\partial w}{\partial t} = \nu \frac{\partial^2 w}{\partial x^2} \tag{22}$$

where $w = w(x,t)$. One of the simplest nonlinear transformation possible here is to use the exponential function as follows

$$w = e^{\alpha\phi} \tag{23}$$

where $\phi = \phi(x,t)$. Solving (23) for ϕ yields

$$\phi = \frac{1}{\alpha}\ln w, \qquad w(x,t) > 0 \tag{24}$$

By the chain rule, we calculate

$$\frac{\partial w}{\partial t} = \alpha\frac{\partial \phi}{\partial t}e^{\alpha\phi} \qquad \frac{\partial w}{\partial x} = \alpha\frac{\partial \phi}{\partial t}e^{\alpha\phi}, \qquad \frac{\partial^2 w}{\partial x^2} = \left[\alpha\frac{\partial^2 \phi}{\partial x^2} + \alpha^2\left(\frac{\partial \phi}{\partial x}\right)^2\right]e^{\alpha\phi}$$

Using the above result, the heat equation (22) can be written as

$$\alpha\frac{\partial \phi}{\partial t}e^{\alpha\phi} = \nu\left[\alpha\frac{\partial^2 \phi}{\partial x^2} + \alpha^2\left(\frac{\partial \phi}{\partial x}\right)^2\right]e^{\alpha\phi}$$

which simplifies to

$$\frac{\partial \phi}{\partial t} = \nu\frac{\partial^2 \phi}{\partial x^2} + \nu\alpha\left(\frac{\partial \phi}{\partial x}\right)^2 \tag{25}$$

We differentiate (25) with respect to x to obtain

$$\frac{\partial^2 \phi}{\partial t \partial x} = \nu\frac{\partial^3 \phi}{\partial x^3} + 2\nu\alpha\frac{\partial \phi}{\partial x}\left(\frac{\partial \phi}{\partial x}\right)^2 \tag{26}$$

If we now set

$$\frac{\partial \phi}{\partial x} = u \tag{27}$$

so that ϕ has the status of a potential function, then the resulting partial differential equation

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial x^2} + 2\nu\alpha u\frac{\partial u}{\partial x}$$

Setting $\alpha = -1/2\nu$ transforms the above equation into the Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

Thus, the Cole-Hopf transformation

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}(-2\nu\ln w) = -2\nu\frac{w_x}{w} \tag{28}$$

reduces the Burgers' equation (21) to the diffusion equation (22).

Let us illustrate the Cole-Hopf transformation with following example. To make things simple, let us first try to obtain the solution of Burgers' equation for a specified solution of the corresponding heat equation

$$\frac{\partial w}{\partial t} = \nu \frac{\partial^2 w}{\partial x^2}$$

with initial condition

$$w(x,0) = b + a\cos kx$$

The solution of the heat equation for the specified initial condition is given by (see (4))

$$w(x,t) = b + ae^{-\nu k^2 t}\cos kx. \tag{29}$$

where $b > |a|$ to ensure that $w(x,t) > 0$ for all time. This solution at various times is plotted in Fig. 6. The solution of the Burgers' equation can now be easily obtained using the Cole-Hopf

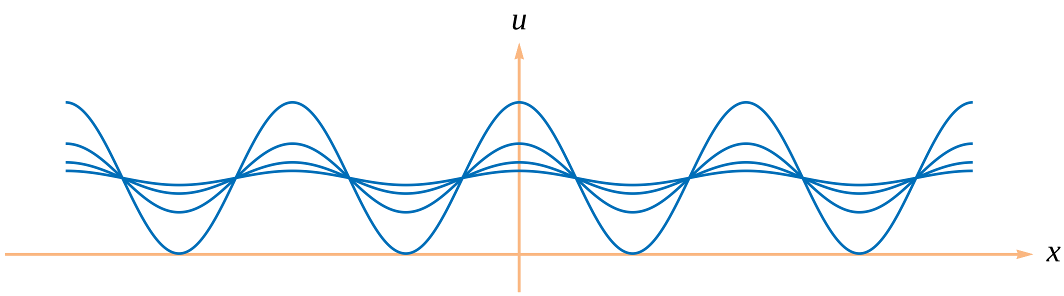

Figure 6: A solution of heat equation.

transformation:

$$u(x,t) = -2\nu \frac{w_x}{w} = \frac{2\nu a k e^{-\nu k^2 t}\sin kx}{b + ae^{-\nu k^2 t}\cos kx} \tag{30}$$

This highly diffusive solution at various times is plotted in Fig. 7.

1.4 General solution of Burgers' equation for in an infinite domain

Let us now consider the general problem of solving the Burgers' equation using Cole-Hopf transformation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \tag{31a}$$

$$u(x,0) = F(x), \qquad -\infty < x < \infty \tag{31b}$$

According to (28), the new variable $w(x,t)$ must initially satisfy

$$u(x,0) = F(x) = -2\nu \frac{w_x(x,0)}{w(x,0)} \tag{32a}$$

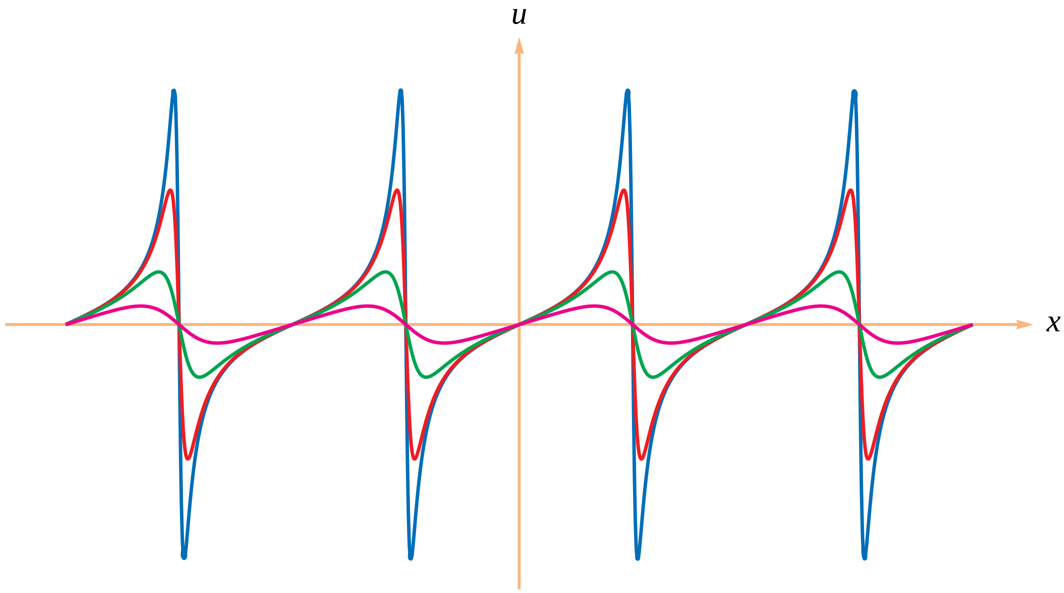

Figure 7: A solution of Burgers' equation.

This is a linear first-order ordinary differential equation for $(x,0)$ and can be written as

$$\frac{1}{w(x,0)}\frac{dw(x,0)}{dx} = -\frac{1}{2\nu}F(x) \tag{32b}$$

The general solution of this equation is

$$w(x,0) = e^{-\frac{1}{2\nu}\int_0^x F(s)ds} = f(x) \tag{33}$$

Note that the lower limit of the integral can be changed from 0 to any other convenient value without affecting the final form of $u(x,t)$ in (28). As a result of the transformation, we only need to solve the linear problem for $w(x,t)$:

$$\frac{\partial w}{\partial t} = \nu\frac{\partial^2 w}{\partial x^2} \tag{34a}$$

$$w(x,0) = f(x), \qquad -\infty < x < \infty \tag{34b}$$

The linear heat equation (34) can be solved using Fourier transform or any other standard techniques. The general solution can be expressed using the heat kernel

$$g(x,t) = \frac{1}{\sqrt{4\pi\nu t}}e^{-\frac{x^2}{4\nu t}} \tag{35}$$

as

$$w(x,t) = \int_{-\infty}^{\infty} f(\eta)g(x-\eta,t)\,d\eta \tag{36a}$$

Therefore, the general solution (34) is given by

$$w(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} f(\eta)\, e^{-\frac{(x-\eta)^2}{4\nu t}}\, d\eta \tag{36b}$$

where $f(\eta)$ is given by (33). It then follows that

$$\frac{\partial w}{\partial x} = \frac{-1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} f(\eta) \frac{(x-\eta)}{2\nu t} e^{-\frac{(x-\eta)^2}{4\nu t}}\, d\eta \tag{37}$$

Therefore, the exact solution of the Burgers' initial value problem is obtained using the transformation (28) in the form

$$u(x,t) = \frac{\displaystyle\int_{-\infty}^{\infty} f(\eta) \frac{(x-\eta)}{t} e^{-\frac{(x-\eta)^2}{4\nu t}}\, d\eta}{\displaystyle\int_{\infty}^{\infty} f(\eta)\, e^{-\frac{(x-\eta)^2}{4\nu t}}\, d\eta} \tag{38}$$

The solution given by (37) is continuous and single valued and for all values of t. It is very hard to give a physical interpretation of this exact solution unless a suitable simple form of $F(x)$ is specified. Even in many problems, an exact evaluation of the integrals involved in (37) is almost a formidable task. It is then necessary to resort to asymptotic methods. We next consider the following example to investigate the formation of discontinuities or shock waves.

The solution to Burgers' equation (38) can be written in a more convenient way as follows. We substitute equation (33) for $f(\eta)$ into (36) to obtain

$$w(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} e^{-G}\, d\eta \tag{39}$$

where

$$G(x,t,\eta) = \frac{1}{2\nu} \int_0^{\eta} F(\theta) d\theta + \frac{(x-\eta)^2}{4\nu t} \tag{40}$$

It then follows that

$$\frac{\partial w}{\partial x} = -\frac{1}{2\nu\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} \frac{x-\eta}{t} e^{-G}\, d\eta \tag{41}$$

Therefore, the alternate form of exact solution of the Burgers' initial value problem is obtained using the transformation (28):

$$u(x,t) = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{x-\eta}{t} e^{-G(x,t,\eta)}\, d\eta}{\displaystyle\int_{-\infty}^{\infty} e^{-G(x,t,\eta)}\, d\eta} \tag{42}$$

It is useful to note that the lower limit in the integral in (40) is arbitrary since it cancels out in (42).

1.4.1 Single hump solution

Find the solution of the Burgers' equation with initial condition

$$u(x,0) = F(x) = A\delta(x) \tag{43}$$

where $\delta(x)$ is the Dirac delta function at the origin and A is a constant.

In the equation (40), the lower limit at 0 of integral for $G(x,t,\eta)$ is problematic, but this difficulty can easily be overcome by selecting a different lower limit of integration. Here we take $-\infty$ as the lower limit. We first calculate,

$$G(x,t,\eta) = \frac{1}{2\nu}\int_{-\infty}^{\eta} A\delta(\theta)d\theta + \frac{(x-\eta)^2}{4\nu t} = \begin{cases} \dfrac{(x-\eta)^2}{4\nu t}, & \eta < 0 \\ \dfrac{(x-\eta)^2}{4\nu t} + \dfrac{A}{2\nu}, & \eta > 0 \end{cases}$$

Substituting the above equation into (42), we can evaluate the upper integral in elementary terms, while the lower integral involves the error function (14.63); after some algebraic manipulations, we obtain (see Whitham [5])

$$u(x,t) = \sqrt{\frac{\nu}{t}}\left[\frac{\left(e^{\frac{A}{2\nu}}-1\right)e^{-\frac{x^2}{4\nu t}}}{\sqrt{\pi}+\left(e^{\frac{A}{2\nu}}-1\right)\sqrt{\frac{\pi}{2}}\,\mathrm{erfc}\left(\frac{x}{\sqrt{4\nu t}}\right)}\right] \tag{44}$$

where the complimentary error function

$$\mathrm{erfc}(z) = 1-\mathrm{erf}(z) = 1-\frac{2}{\sqrt{\pi}}\int_0^z e^{-r^2}dr = \frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-r^2}dr. \tag{45}$$

This is a case of an initial single hump propagating in the x direction and simultaneously diffuses out as shown in Fig. 8. The effect of the nonlinear advection term is to steepen the front side of hump as it propagates. Eventually the triangular wave spreads out as the diffusion progresses.

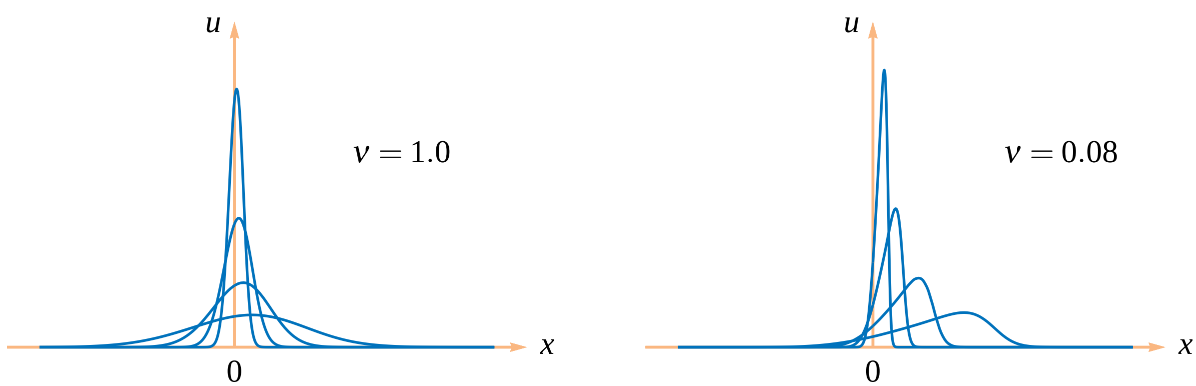

Figure 8: Single hump solution of Burgers' equation for high (left) and low (right) viscosity.

Using method of similarity transformation, we could obtain the form of the solutions as

$$u(x,t) = \sqrt{\frac{\nu}{t}}\, g\left(\frac{x}{\sqrt{\nu t}}, \frac{A}{\nu}\right) \tag{46}$$

which has the same form as (44). Note that there is no separate length and time with which to scale x and t separately. Define the similarity variable

$$z = \frac{x}{\sqrt{4\nu t}}$$

and the Reynolds number

$$R = \frac{A}{2\nu}$$

the solution (44) can be written in dimensionless form as

$$v(z,t) = \frac{\left(e^R - 1\right) e^{-z^2}}{\sqrt{\pi} + \left(e^R - 1\right)\sqrt{\frac{\pi}{2}}\,\mathrm{erfc}(z)} \tag{47}$$

where $v = u/\sqrt{\nu/t}$ is the dimensionless velocity. The form of the triangular wave solution for different R is shown in Fig. 9.

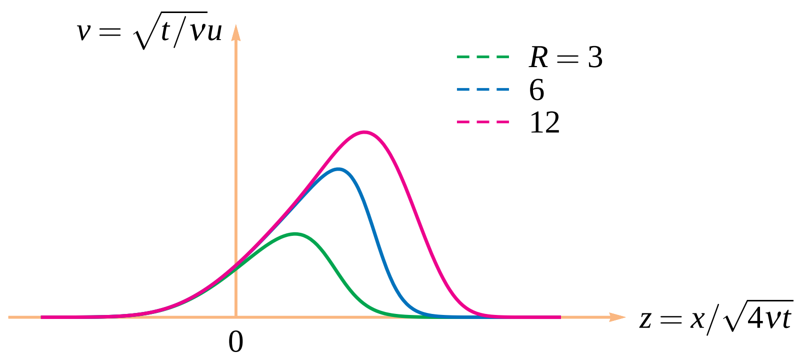

Figure 9: Triangular wave (single hump) solution of Burgers' equation – dimensionless form.

For this problem, two limiting cases can be brought out. The first is the case when $\nu \to \infty$ and second is $\nu \to 0$. In the first case we expect diffusion term to dominate than that of nonlinear advection. Therefore, in the limit as $\nu \to \infty$, we have

$$\mathrm{erfc}\left(\frac{x}{\sqrt{4\nu t}}\right) \to 0 \qquad \text{and} \qquad e^{\frac{A}{2\nu}} \to 1 + \frac{A}{2\nu}$$

the solution (44) tends to the limiting value

$$u(x,t) = \frac{A}{\sqrt{4\pi\nu t}} e^{-\frac{x^2}{4\nu t}} \tag{48}$$

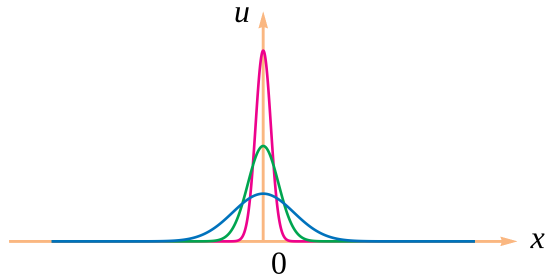

Figure 10: Solution of linear heat equation

which is infinitely differentiable for positive t. This is the fundamental solution of the heat equation (35). For all time t, the hump shaped solution remains Gaussian as shown in Fig. 10. The height of hump decreases inversely with $\sqrt{\nu t}$, whereas the width of the hump increases with $\sqrt{\nu t}$.

In the second case as $\nu \to 0$, we expect the nonlinear advection term to dominate than that of linear diffusion. Therefore, in the limit as $\nu \to 0$, it possible to show that (44) reduces to the shock wave solution of inviscid Burgers' equation

$$u(x,t) \sim \begin{cases} \dfrac{x}{t}, & 0 < x < \sqrt{2At} \\ 0, & \text{otherwise.} \end{cases} \tag{49}$$

which is the asymptotic solution of Burgers' equation with single hump as the initial profile (see Whitham [5]). This result represents a shock wave at $x = \sqrt{2At}$, and the shock speed $U = \sqrt{A/2t}$. The solution jumps from 0 to $\sqrt{A/2t}$.

1.4.2 Asymptotic behaviour of Burgers' Solution

The behavior of the exact solution of Burgers' equation (42) can be obtained for $\nu \to 0$ while x, t and $F(x)$ are held fixed. As the dominant contributions to the integrals in (42) come from the neighborhood of the stationary points of G. A stationary point is where

$$\frac{\partial G}{\partial \eta} = F(\theta) - \frac{x-\eta}{t} = 0. \tag{50}$$

Suppose that $\eta = \xi(x,t)$ is a solution of (49) representing a stationary point. The contribution from the neighborhood of a stationary point, $\eta = \xi$ an integral

$$\int_{-\infty}^{\infty} g(\eta)\, e^{-G(\eta)} d\eta$$

is

$$g(\xi)\sqrt{\frac{2\pi}{|G''(\xi)|}}\, e^{-G(\xi)}.$$

This is the standard formula of the method of steepest descents. Suppose first that there is only one stationary point $\xi(x,t)$ which satisfies (49). Then

$$\int_{-\infty}^{\infty} \frac{x-\eta}{t} e^{-G(x,t,\eta)}\, d\eta \sim \frac{x-\xi}{t}\sqrt{\frac{2\pi}{|G''(\xi)|}}\, e^{-G(\xi)}, \tag{51}$$

$$\int_{-\infty}^{\infty} e^{-G(x,t,\eta)}\, d\eta \sim \sqrt{\frac{2\pi}{|G''(\xi)|}}\, e^{-G(\xi)}, \tag{52}$$

and in (42) we have

$$u \sim \frac{x-\xi}{t} \tag{53}$$

where $\xi(x,t)$ is defined by (50). This asymptotic solution may be rewritten

$$\left.\begin{aligned} u(x,t) &= F(\xi) \\ \xi &= x - F(\xi)t. \end{aligned}\right\} \tag{54}$$

This is identical with the solution of the Burgers' equation without the diffusion term $\nu = 0$. Here, the stationary point ξ corresponds to the characteristic variable in the context of the first-order, quasilinear equation. Although the exact solution of Burgers' equation is a single-valued and continuous function for all time t, the asymptotic solution (54) exhibits instability. It has already been shown that this progressively distorts itself and becomes multiple-valued after sufficiently long time. Eventually, it would break with the development of discontinuity as a shock wave. When this state is reached, there will be two stationary pints of (50), and then, some modification is required to complete the asymptotic analysis.

1.4.3 Discontinuous initial profile

Find the solution of the Burgers' equation with piece-wise initial condition

$$u(x,0) = F(x) = aH(-x) + bH(x) = \begin{cases} a, & x < 0 \\ b, & x > 0 \end{cases} \tag{55}$$

where $H(x)$ is the Heaviside function. We assume that $a > b$, which would correspond to a shock wave in the inviscid limit $\nu = 0$.

We first calculate,

$$G(x,t,\eta) = \frac{1}{2\nu}\int_0^{\eta} F(\theta)\, d\theta + \frac{(x-\eta)^2}{4\nu t} = \begin{cases} \dfrac{(x-\eta)^2}{4\nu t} - \dfrac{a}{2\nu}\eta, & \eta < 0 \\ \dfrac{(x-\eta)^2}{4\nu t} - \dfrac{b}{2\nu}\eta, & \eta > 0 \end{cases}$$

Substituting the above equation in (42) and integration independently from $-\infty$ to 0 and from 0 to ∞, we obtain

$$u(x,t) = \frac{e^{\frac{b-a}{2\nu}(x-ct)} a\,\mathrm{erfc}\left(\frac{x-bt}{\sqrt{4\nu t}}\right) + b\,\mathrm{erfc}\left(\frac{x-at}{\sqrt{4\nu t}}\right)}{e^{\frac{b-a}{2\nu}(x-ct)}\mathrm{erfc}\left(\frac{x-bt}{\sqrt{4\nu t}}\right) + \mathrm{erfc}\left(\frac{x-at}{\sqrt{4\nu t}}\right)} \tag{56}$$

where

$$c = \frac{a+b}{2}$$

and erfc(.) the complimentary error function. The solution to Burgers' equation (56) can be written in a more convenient way as follows:

$$u(x,t) = a - \frac{a-b}{1+q(x,t)e^{-\frac{a-b}{2\nu}(x-ct)}} \tag{57a}$$

where

$$q(x,t) = \frac{\operatorname{erfc}\left(\frac{x-bt}{\sqrt{4\nu t}}\right)}{\operatorname{erfc}\left(\frac{x-at}{\sqrt{4\nu t}}\right)}.$$

For the special case of $b=0$, the solution (57a) simplifies to

$$u(x,t) = a - \frac{a}{1+q(x,t)e^{-\frac{c}{\nu}(x-ct)}} \tag{57b}$$

where

$$q(x,t) = \frac{\operatorname{erfc}\left(\frac{x}{\sqrt{4\nu t}}\right)}{\operatorname{erfc}\left(\frac{x-at}{\sqrt{4\nu t}}\right)}.$$

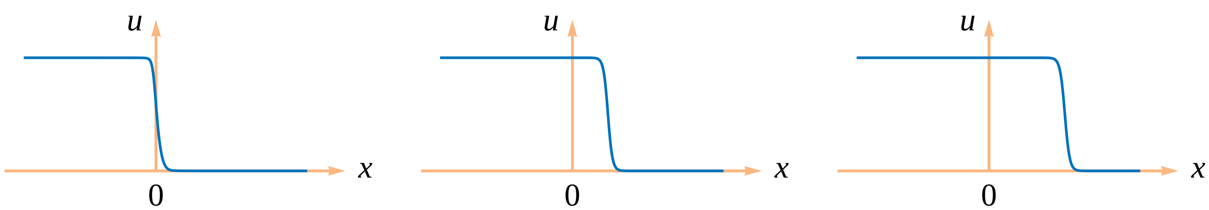

Figure 11: Shock wave solution (57) of Burgers' equation for low viscosity.

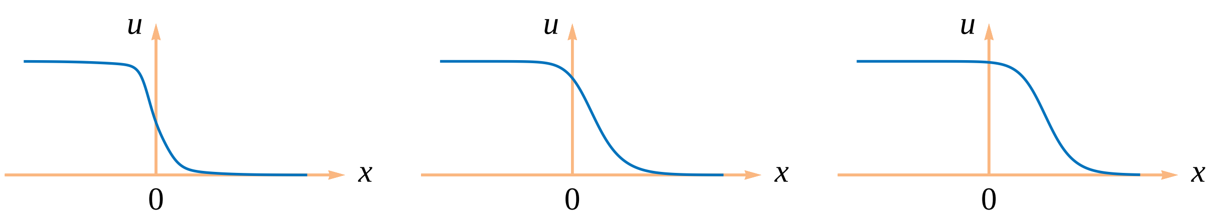

Figure 12: Shock wave solution (57) of Burgers' equation for high viscosity.

The solution in (57) is plotted in figures 11 and 12 at times $t = 0.05, 0.7, 1.4$. Note that the sharp discontinuity is immediately smoothed, and the solution rapidly settles into the form of a continuously varying transition layer between the two discontinuous velocity. The smoothing effect is more for larger viscosity.

Example: The N-wave solution

An N-wave solution is an exact solution of Burgers' equation which is created if a rarefaction-compression pulse is specified as the initial waveform. Note that, for the single hump problem, we could have taken the solution of diffusion equation (22) corresponding to an initial step function. To obtain an N-wave for u, we choose the source solution of the heat equation (22) of the form

$$w(x,t) = 1 + \sqrt{\frac{\tau}{t}}\, e^{-\frac{x^2}{4\nu t}} \tag{58}$$

where τ is a constant. Using Cole-Hopf transformation (28), we obtain

$$u(x,t) = -2\nu \frac{w_x}{w} = \frac{x}{t} \frac{\sqrt{\tau/t}\, e^{-\frac{x^2}{4\nu t}}}{1 + \sqrt{\tau/t}\, e^{-\frac{x^2}{4\nu t}}} \tag{59a}$$

$$= \frac{x}{t}\left[1 + \sqrt{\tau/t}\, e^{-\frac{x^2}{4\nu t}}\right]^{-1}. \tag{59b}$$

For any time $t > 0$, this solution is shown in Fig. 11 and has the inverted N-shaped form with a positive and negative phases. Note that since w has the behavior of a delta-function as $t \to 0$, this cannot be interpreted as an initial value problem on u.

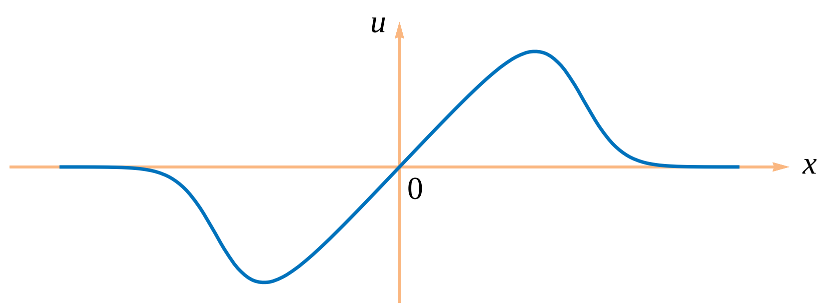

Figure 13: N-wave solution of the Burgers' equation

The area under the positive phase of the wave profile is given by

$$\int_0^{\infty} u(x,t)\, dx = -2\nu \left[\ln w(x,t)\right]_0^{\infty} = 2\nu \ln(1 + \sqrt{\tau/t}). \tag{60}$$

The magnitude of the area of the negative phase is the same. As we can see, the area of the positive phase tends to zero as $t \to \infty$. Let the value of area given by (60) at the initial time t_0 is denoted by A. We now introduce an initial Reynolds number

$$R_0 = \frac{A}{2\nu} = \ln(1 + \sqrt{\tau/t_0}) \tag{61}$$

and the effective Reynolds number at ant time t,

$$R(t) = \frac{1}{2\nu} \int_0^{\infty} u(x,t)\, dx = \ln(1 + \sqrt{\tau/t}) \tag{62}$$

$R(t)$ tends to zero as $t \to \infty$. An expression for the constant τ can be obtained by rearranging (61),

$$\tau = t_0 \left(e^{R_0} - 1\right)^{1/2}.$$

With this, the solution (59) can be written as

$$u(x,t) = \frac{x}{t} \left[1 + \sqrt{\frac{t}{t_0}} \, \frac{e^{-\frac{Rx^2}{2At}}}{e^{R_0 - 1}} \right]^{-1}. \tag{63}$$

For the case of $R \ll 1$, we have $e^{R_0 - 1} \sim e_0^R$, therefore (63) reduces to the form

$$u(x,t) \sim \frac{x}{t} \left[1 + \sqrt{\frac{t}{t_0}} \, e^{R_0 \left(\frac{x^2}{2At} - 1 \right)} \right]^{-1} \tag{64}$$

for all x and t.

In the limit, as $R_0 \to \infty$ with t fixed, (64) gives the shock wave solution of inviscid Burgers' equation (2) in the form

$$u(x,t) \sim \begin{cases} \dfrac{x}{t}, & -\sqrt{2At} < x < \sqrt{2At} \\ 0, & |x| > \sqrt{2At}. \end{cases} \tag{65}$$

Note that, in the limit as $t \to \infty$, for fixed values of ν and τ, solution (59a) takes the form

$$u(x,t) \sim \frac{x}{t} \sqrt{\frac{\tau}{t}} \, e^{\frac{-x^2}{4\nu t}} \tag{66}$$

This corresponds to the dipole solution of the heat equation (3). The diffusion dominates the nonlinear term in the final decay. It should be remembered, though, that this final period of decay is for extremely large times and the inviscid theory is adequate in this case.

References

1. Benton, E. R. and G. W. Platzman, *A table of solutions of one-dimensional Burgers' equation*, Quart. Appl. Math. **30**, 195-212 (1972).

2. Debnath, L., *Nonlinear Partial Differential Equations for Scientists and Engineers*, 3^{rd} ed., Birkhauser (2011).

3. Olver, P. J., *Introduction to Partial Differential Equations*, Springer (2014).

4. Renardy, M. and R. C. Rogers, *An Introduction to Partial Differential Equations*, 2^{nd} ed., Springer (2004).

5. Whitham, G. B., *Linear and Nonlinear Waves*, John Wiley (1974).