Eigenvalue Problem

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The method of separation variables for solving the heat equation often leads transformation of PDEs to ODEs. The ODE thus obtained will be in the form of an eigenvalue problem. Four types of eigenvalue problems are commonly encountered as discussed below.

1 The First (Dirichlet) Eigenvalue Problem

Consider the second order differential equation

$$\frac{d^2X}{dx^2} + \lambda X = 0, \qquad 0 < x < L \tag{1a}$$

with Dirichlet type boundary conditions

$$X(0) = 0, \qquad X(L) = 0 \tag{1b}$$

The problem here is to find the values of λ and the nontrivial solutions $X(x)$ of (1). All the conditions in this problem are linear and homogeneous, and so any nonzero constant times a nontrivial solution $X(x)$ is essentially the same as $X(x)$. This problem is called an *eigenvalue problem*. The Dirichlet eigenvalue problem involves the determination of a solution $X(x)$ of (1) in a domain $[0, L]$ for some λ that satisfies the boundary conditions $X(0) = X(L) = 0$. This is a special case of more general problem called *Sturm-Liouville problem*. The possible solutions of (1) fall into the following three cases:

Case 1 ($\lambda = 0$)
If $\lambda = 0$, the differential equation (1a) reduces to

$$X'' = 0$$

and its general solution is

$$X(x) = c_1 + c_2 x$$

where c_1 and c_2 are constants. The boundary condition $X(0) = 0$ requires that $c_1 = 0$. So $X(x) = c_2 x$. Further, the boundary condition $X(L) = 0$ requires that $c_2 = 0$. So the eigenvalue problem (1) has only trivial solution $X(x) \equiv 0$ if $\lambda = 0$ and hence $\lambda = 0$ is not an eigenvalue.

Case 2 ($\lambda < 0$)

If $\lambda < 0$, we write $\lambda = -\omega^2$ $(\omega > 0)$ and the differential equation (1a) becomes

$$X'' - \omega^2 X = 0$$

This equation has the general solution

$$X(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}$$

The condition $X(0) = 0$ requires that

$$c_2 = -c_1$$

Hence

$$X(x) = c_1 \left(e^{\omega x} - e^{-\omega x}\right) = 2c_1 \sinh \omega x$$

But the condition $X(L) = 0$ requires that

$$2c_1 \sinh \omega L = 0$$

This implies that $c_1 = 0$. Again we see that the eigenvalue problem (1) has only trivial solution $X(x) \equiv 0$ if $\lambda < 0$ and hence negative values of λ are not eigenvalues.

Case 3 ($\lambda > 0$)

If $\lambda > 0$, we write $\lambda = \omega^2$ $(\omega > 0)$ and the differential equation becomes

$$X'' + \omega^2 X = 0$$

This equation has the general solution

$$X(x) = c_1 \cos \omega x + c_2 \sin \omega x$$

The condition $X(0) = 0$ requires that $c_1 = 0$. Hence

$$X(x) = c_2 \sin \omega x$$

Further, the boundary condition $X(L) = 0$ requires that

$$c_2 \sin \omega L = 0$$

We see that $c_2 = 0$ results in trivial solution of (1). Hence for nontrivial solution ω must be positive roots of the equation

$$\sin \omega L = 0$$

from which we have

$$\omega = \frac{n\pi}{L} \qquad (n = 1, 2, \cdots)$$

So, except for the constant factor c_2

$$X(x) = \sin \frac{n\pi x}{L} \qquad (n = 1, 2, \cdots)$$

and the corresponding values of λ are

$$\lambda = \omega^2 = \left(\frac{n\pi}{L}\right)^2 \qquad (n = 1, 2, \cdots)$$

Thus the discrete values

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad (n = 1, 2, \cdots) \tag{2}$$

of λ for which problem (1) has nontrivial solutions are called *eigenvalues* of that problem, and the solutions

$$X_n(x) = \sin\frac{n\pi x}{L} \qquad (n = 1, 2, \cdots) \tag{3}$$

are the corresponding *eigenfunctions*.

2 The Second (Neumann) Eigenvalue Problem

The second eigenvalue problem (Sturm-Liouville problem) that comes up frequently is

$$\frac{d^2X}{dx^2} + \lambda X = 0, \qquad 0 < x < L \tag{4a}$$

with Neumann type boundary conditions

$$X'(0) = 0, \qquad X'(L) = 0 \tag{4b}$$

This problem is called a *Neumann eigenvalue problem*. By the Neumann eigenvalue problem we mean the determination of a solution $X(x)$ of (4) in a domain $[0, L]$ for some λ that satisfies the boundary conditions $X'(0) = X'(L) = 0$. The possible solutions of (4) fall into the following three cases:

Case 1 ($\lambda = 0$)
If $\lambda = 0$, the differential equation (4a) has the general solution

$$X(x) = c_1 + c_2 x$$

where c_1 and c_2 are constants. Application of boundary conditions requires the first derivative of X:

$$X' = c_2$$

The boundary condition $X'(0) = 0$ requires that $c_2 = 0$. So

$$X(x) = c_1$$

Further, the boundary condition $X'(L) = 0$ automatically satisfied. So the eigenvalue problem (4) has a nontrivial solution if $\lambda = 0$ and hence $\lambda_0 = 0$ is an eigenvalue.

Case 2 ($\lambda < 0$)
If $\lambda < 0$, we write $\lambda = -\omega^2\,(\omega > 0)$ and the differential equation (4a) becomes

$$X'' - \omega^2 X = 0$$

This equation has the general solution

$$X(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}$$

and its derivative is given by

$$X' = c_1 \omega e^{\omega x} - c_2 \omega e^{-\omega x}$$

The condition $X'(0) = 0$ requires that

$$c_2 = c_1$$

Hence

$$X(x) = c_1\left(e^{\omega x} + e^{-\omega x}\right) = 2c_1 \cosh \omega x \qquad \text{and} \qquad X' = 2c_1 \omega \sinh \omega x$$

But the condition $X'(L) = 0$ requires that

$$2c_1 \omega \sinh \omega L = 0$$

This implies that $c_1 = 0$ and thus the eigenvalue problem (4) has only trivial solution $X(x) \equiv 0$ if $\lambda < 0$ and hence the eigenvalues cannot be negative.

Case 3 ($\lambda > 0$)
If $\lambda > 0$, we write $\lambda = \omega^2\,(\omega > 0)$ and the differential equation becomes

$$X'' + \omega^2 X = 0$$

This equation has the general solution

$$X(x) = c_1 \cos \omega x + c_2 \sin \omega x$$

and its derivative is given by

$$X' = -c_1 \omega \sin \omega x + c_2 \omega \cos \omega x$$

The condition $X'(0) = 0$ requires that $c_2 = 0$. Hence

$$X(x) = c_1 \cos \omega x \qquad \text{and} \qquad X' = -c_1 \omega \sin \omega x$$

The boundary condition $X'(L) = 0$ requires that

$$c_1 \omega \sin \omega L = 0$$

We see that $c_1 = 0$ results in trivial solution of (4). Hence for nontrivial solution ω must be positive roots of the equation

$$\sin \omega L = 0$$

from which we have

$$\omega = \frac{n\pi}{L} \qquad (n = 1, 2, \cdots)$$

So, except for the constant factor c_1

$$X(x) = \cos\frac{n\pi x}{L} \qquad (n = 1, 2, \cdots)$$

and the corresponding values of λ are

$$\lambda = \omega^2 = \left(\frac{n\pi}{L}\right)^2 \qquad (n = 1, 2, \cdots)$$

Thus the discrete eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad (n = 1, 2, \cdots) \tag{5}$$

and the corresponding eigenfunctions are given by

$$X_n(x) = \cos\frac{n\pi x}{L} \qquad (n = 1, 2, \cdots) \tag{6}$$

3 The Third (Mixed Dirichlet-Neumann) Eigenvalue Problem

The first version

The third eigenvalue problem (Sturm-Liouville problem) that comes up frequently is

$$\frac{d^2X}{dx^2} + \lambda X = 0, \qquad 0 < x < L \tag{7a}$$

with the boundary conditions of the type

$$X(0) = 0, \qquad X'(L) = 0 \tag{7b}$$

where the differential equation and the domain is same as in problem (1) and the possible solutions of (7) fall into the following three cases:

Case 1 ($\lambda = 0$)
If $\lambda = 0$, the differential equation (7a) has the general solution

$$X(x) = c_1 + c_2 x$$

where c_1 and c_2 are constants. The boundary condition $X(0) = 0$ requires that $c_1 = 0$. So

$$X(x) = c_2 x \qquad \text{and} \qquad X' = c_2$$

The boundary condition $X'(L) = 0$ requires that $c_2 = 0$. So the eigenvalue problem (7) has only a trivial solution if $\lambda = 0$ and hence $\lambda = 0$ is not an eigenvalue.

Case 2 ($\lambda < 0$)
If $\lambda < 0$, we write $\lambda = -\omega^2\,(\omega > 0)$ and the differential equation (7a) becomes

$$X'' - \omega^2 X = 0$$

This equation has the general solution

$$X(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}$$

and its derivative is given by

$$X' = c_1 \omega e^{\omega x} - c_2 \omega e^{-\omega x}$$

The condition $X(0) = 0$ requires that

$$c_2 = -c_1$$

Hence

$$X(x) = c_1\left(e^{\omega x} - e^{-\omega x}\right) = 2c_1 \sinh \omega x \qquad \text{and} \qquad X' = 2c_1 \omega \cosh \omega x$$

But the condition $X'(L) = 0$ requires that

$$2c_1 \omega \cosh \omega L = 0$$

This implies that $c_1 = 0$ and thus the eigenvalue problem (7) has only trivial solution $X(x) \equiv 0$ if $\lambda < 0$ and hence the eigenvalues cannot be negative.

Case 3 ($\lambda > 0$)
If $\lambda > 0$, we write $\lambda = \omega^2\,(\omega > 0)$ and the differential equation becomes

$$X'' + \omega^2 X = 0$$

This equation has the general solution

$$X(x) = c_1 \cos \omega x + c_2 \sin \omega x$$

The condition $X(0) = 0$ requires that $c_1 = 0$. Hence

$$X(x) = c_2 \sin \omega x \qquad \text{and} \qquad X' = c_2 \omega \cos \omega x$$

The boundary condition $X'(L) = 0$ requires that

$$c_2 \omega \cos \omega L = 0$$

We see that $c_2 = 0$ results in trivial solution of (7). Hence for nontrivial solution ω must be positive roots of the equation

$$\cos \omega L = 0$$

from which we have

$$\omega = \frac{\left(n - \frac{1}{2}\right)\pi}{L} \qquad (n = 1, 2, \cdots)$$

So, except for the constant factor c_2

$$X(x) = \sin \frac{\left(n - \frac{1}{2}\right) \pi x}{L} \qquad (n = 1, 2, \cdots)$$

and the corresponding values of λ are

$$\lambda = \omega^2 = \left[\frac{\left(n - \frac{1}{2}\right) \pi}{L} \right]^2 \qquad (n = 1, 2, \cdots)$$

Thus the discrete eigenvalues are

$$\lambda_n = \left[\frac{\left(n - \frac{1}{2}\right) \pi}{L} \right]^2 \qquad (n = 1, 2, \cdots) \tag{8a}$$

and the corresponding eigenfunctions are given by

$$X_n(x) = \sin \frac{\left(n - \frac{1}{2}\right) \pi x}{L} \qquad (n = 1, 2, \cdots) \tag{8b}$$

The second version

This a variation of the third eigenvalue problem (Sturm-Liouville problem) of the first version.

$$\frac{d^2X}{dx^2} + \lambda X = 0, \qquad 0 < x < L \tag{9a}$$

with the boundary conditions of the type

$$X'(0) = 0, \qquad X(L) = 0 \tag{9b}$$

The possible solutions of (9) fall into the following three cases:

Case 1 ($\lambda = 0$)
If $\lambda = 0$, the differential equation (9a) has the general solution

$$X(x) = c_1 + c_2 x$$

where c_1 and c_2 are constants. Application of boundary conditions requires the first derivative of X:

$$X' = c_2$$

The boundary condition $X'(0) = 0$ requires that $c_2 = 0$. So

$$X(x) = c_1$$

The boundary condition $X(L) = 0$ requires that $c_1 = 0$. So the eigenvalue problem (9) has only a trivial solution if $\lambda = 0$ and hence $\lambda = 0$ is not an eigenvalue.

Case 2 ($\lambda < 0$)
If $\lambda < 0$, we write $\lambda = -\omega^2\,(\omega > 0)$ and the differential equation (9a) becomes

$$X'' - \omega^2 X = 0$$

This equation has the general solution

$$X(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}$$

and its derivative is given by

$$X' = c_1 \omega e^{\omega x} - c_2 \omega e^{-\omega x}$$

The condition $X'(0) = 0$ requires that

$$c_2 = c_1$$

Hence

$$X(x) = c_1\left(e^{\omega x} + e^{-\omega x}\right) = 2c_1 \cosh \omega x$$

But the condition $X(L) = 0$ requires that

$$2c_1 \cosh \omega L = 0$$

This implies that $c_1 = 0$ and thus the eigenvalue problem (9) has only trivial solution $X(x) \equiv 0$ if $\lambda < 0$ and hence the eigenvalues cannot be negative.

Case 3 ($\lambda > 0$)
If $\lambda > 0$, we write $\lambda = \omega^2\,(\omega > 0)$ and the differential equation becomes

$$X'' + \omega^2 X = 0$$

This equation has the general solution

$$X(x) = c_1 \cos \omega x + c_2 \sin \omega x$$

and its derivative is given by

$$X' = -c_1 \omega \sin \omega x + c_2 \omega \cos \omega x$$

The condition $X'(0) = 0$ requires that $c_2 = 0$. Hence

$$X(x) = c_1 \cos \omega x$$

The boundary condition $X(L) = 0$ requires that

$$c_1 \cos \omega L = 0$$

We see that $c_1 = 0$ results in trivial solution of (9). Hence for nontrivial solution ω must be positive roots of the equation

$$\cos \omega L = 0$$

from which we have

$$\omega = \frac{\left(n-\frac{1}{2}\right)\pi}{L} \qquad (n = 1, 2, \cdots)$$

So, except for the constant factor c_1

$$X(x) = \cos\frac{\left(n-\frac{1}{2}\right)\pi x}{L} \qquad (n = 1, 2, \cdots)$$

and the corresponding values of λ are

$$\lambda = \omega^2 = \left[\frac{\left(n-\frac{1}{2}\right)\pi}{L}\right]^2 \qquad (n = 1, 2, \cdots)$$

Thus the discrete eigenvalues are

$$\lambda_n = \left[\frac{\left(n-\frac{1}{2}\right)\pi}{L}\right]^2 \qquad (n = 1, 2, \cdots) \tag{10a}$$

and the corresponding eigenfunctions are given by

$$X_n(x) = \cos\frac{\left(n-\frac{1}{2}\right)\pi x}{L} \qquad (n = 1, 2, \cdots) \tag{10b}$$

4 The Fourth (Periodic) Eigenvalue Problem

Consider the second order differential equation

$$\frac{d^2X}{dx^2} + \lambda X = 0, \qquad 0 < x < L \tag{11a}$$

with the boundary conditions

$$X(0) = X(L), \qquad X'(0) = X'(L) \tag{11b}$$

This problem is called a *periodic eigenvalue problem*. By the periodic eigenvalue problem we mean the determination of a solution $X(x)$ of (11) in a periodic domain $[0, L]$ for some λ that satisfies the periodic boundary conditions (11b). This is a special case of more general problem called *periodic Sturm-Liouville problem*. The possible solutions of (11) fall into the following three cases:

Case 1 ($\lambda = 0$)
If $\lambda = 0$, the differential equation (11a) has the general solution

$$X(x) = c_1 + c_2 x$$

where c_1 and c_2 are constants. The boundary condition $X(0) = X(L)$ requires that $c_2 = 0$ and $X(x) = c_1$. So the eigenvalue problem (11) has a nontrivial solution if $\lambda = 0$ and hence $\lambda_0 = 0$ is an eigenvalue with a corresponding eigenfunction 1.

Case 2 ($\lambda < 0$)
If $\lambda < 0$, we write $\lambda = -\omega^2\,(\omega > 0)$ and the differential equation (11a) becomes

$$X'' - \omega^2 X = 0$$

This equation has the general solution

$$X(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}$$

The condition $X(0) = X(L)$ gives

$$c_1 + c_2 = c_1 e^{\omega L} + c_2 e^{-\omega L}$$

and condition $X'(0) = X'(L)$ gives

$$c_1 - c_2 = c_1 e^{\omega L} - c_2 e^{-\omega L}$$

On simplification the above two equations gives

$$e^{\omega L} = e^{-\omega L}$$

which cannot be satisfied by any nonzero values of ω and hence $\lambda < 0$ are not eigenvalues of (11).

Case 3 ($\lambda > 0$)
If $\lambda > 0$, we write $\lambda = \omega^2\,(\omega > 0)$ and the differential equation becomes

$$X'' + \omega^2 X = 0$$

This equation has the general solution

$$X(x) = c_1 \cos \omega x + c_2 \sin \omega x$$

The condition $X(0) = X(L)$ gives

$$c_1 = c_1 \cos \omega L + c_2 \sin \omega L$$

and condition $X'(0) = X'(L)$ gives

$$c_2 = c_2 \cos \omega L - c_1 \sin \omega L$$

For nonzero values of c_1 and c_2, on simplification the above two equations yields

$$\sin \omega L = 0 \qquad \text{and} \qquad \cos \omega L = 1$$

both of which are satisfied if

$$\omega = \frac{2n\pi}{L} \qquad (n = 1, 2, \cdots)$$

Therefore the discrete eigenvalues are

$$\lambda_n = \left(\frac{2n\pi}{L}\right)^2 \qquad (n = 1, 2, \cdots)$$

and we obtain two linearly independent eigenfunctions

$$\cos\frac{2n\pi x}{L} \quad \text{and} \quad \sin\frac{2n\pi x}{L} \qquad (n = 1, 2, \cdots)$$

corresponding to the same eigenvalue λ_n.

In conclusion, the solution of the periodic eigenvalue problem (11) is the following infinite sequence of eigenvalues

$$\lambda_0 = 0, \qquad \lambda_n = \left(\frac{2n\pi}{L}\right)^2 \qquad (n = 1, 2, \cdots) \tag{12a}$$

and the corresponding eigenfunctions

$$X_0(x) = 1, \qquad X_n(x) = \cos\frac{2n\pi x}{L}, \sin\frac{2n\pi x}{L} \qquad (n = 1, 2, \cdots) \tag{12b}$$

It may be noted that for the eigenvalue problems previously considered (Dirichlet, Neumann, and mixed Dirichlet-Neumann), we saw that there exists only one linearly independent eigenfunction X_n corresponding to the eigenvalue λ_n, which is called an eigenvalue of multiplicity one (or a simple eigenvalue). On the contrary, for the periodic eigenvalue problem, the eigenfunctions $\cos(2n\pi x/L)$ and $\sin(2n\pi x/L)$ correspond to the same eigenvalue $(2n\pi/L)^2$. Thus, this eigenvalue is of multiplicity two.

Summary

We give a summary of the results of various eigenvalue problems:

1. First (Dirichlet) eigenvalue problem:

$$X(0) = X(L) = 0 \qquad \Longrightarrow \qquad X_n(x) = \sin\frac{n\pi x}{L} \qquad (n = 1, 2, \cdots)$$

2. Second (Neumann) eigenvalue problem:

$$X'(0) = X'(L) = 0 \qquad \Longrightarrow \qquad X_n(x) = \cos\frac{n\pi x}{L} \qquad (n = 0, 1, 2, \cdots)$$

3. Third (mixed Dirichlet-Neumann) eigenvalue problem I:

$$X(0) = X'(L) = 0 \qquad \Longrightarrow \qquad X_n(x) = \sin\frac{\left(n-\frac{1}{2}\right)\pi x}{L} \qquad (n = 1, 2, \cdots)$$

4. Third (mixed Dirichlet-Neumann) eigenvalue problem II:

$$X'(0) = X(L) = 0 \qquad \Longrightarrow \qquad X_n(x) = \cos\frac{\left(n-\frac{1}{2}\right)\pi x}{L} \qquad (n = 1, 2, \cdots)$$

5. Fourth (periodic) eigenvalue problem:

$$X(0) = X(L),\ X'(0) = X'(L) \qquad \Longrightarrow \qquad X_0 = 1,\ X_n(x) = \cos\frac{2n\pi x}{L}, \sin\frac{2n\pi x}{L} \qquad (n = 1, 2, \cdots)$$