Second-Order Wave Equation

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1 Introduction

The classical wave equation is a second-order linear partial differential equation given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right). \tag{1}$$

It models many physical phenomena in nature including, the vibrations of a string in one dimension, the vibrations of a thin membrane in two dimensions, or the pressure vibrations of an acoustic wave in three dimensions. The constant coefficient c gives the speed of propagation of wave. In one dimension, the wave equation reduces to

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{2}$$

The dependent variable u in equation (1) may represent the perturbation velocity (or surface height) for small-amplitude water waves or the perturbation velocity or density for small-amplitude disturbances in a one-dimensional compressible flow. A third interpretation has u representing either the lateral or axial displacement of a vibrating string in tension in the limit of small-amplitude oscillations.

2 d'Alembert's Solution in Infinite Domain

We shall consider the following Cauchy problem of an infinite domain with two initial conditions.

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= F(x) \qquad -\infty < x < \infty \\ u_t(x,0) &= G(x) \qquad -\infty < x < \infty \end{aligned} \tag{3}$$

A justification for imposing two initial conditions (rather than one, as in the diffusion equation) is to argue that the PDE in (3) is second order in t and therefore requires two conditions in order to define a solution uniquely. Using physical reasoning, for example, for the vibrating string, we would argue that in order to define the state of a dynamical system, we must initially specify both the displacement and the velocity.

Like heat equation and Laplace equation, the solution of second-order wave equation can also be obtained using the standard method of separation of variables or Fourier transform. However, here we consider a different approach, the d'Alembert's solution of the wave equation, which is more suitable if the domain is infinite. In this method, a canonical form of the wave equation (3) is first obtained using a suitable transformation. The canonical form enable us to easily integrate the equation to obtain the general solution. Below we give a brief description of the solution method.

We can factor the linear differential operators of (3) to give[1]

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u = 0. \tag{4}$$

We consider the following transformation of variables

$$\xi = x - ct \qquad \eta = x + ct \tag{5}$$

where ξ and η are called *characteristic variables*. Equation (5) represents two families of characteristics for the wave equation. To express (3) in terms of ξ and η, we use the chain rule to evaluate the terms in the equation

$$\begin{aligned}\frac{\partial}{\partial t} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t} = -c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\\ \frac{\partial}{\partial x} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\end{aligned}$$

Using the above set of relations, (4) can be written as

$$\left[c\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right) + c\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\right]\left[c\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right) - c\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\right]u = 0$$

which on simplification,

$$2c\left[\frac{\partial}{\partial \eta}\right]2c\left[-\frac{\partial}{\partial \xi}\right]u = 0$$

or

$$\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) = 0. \tag{6}$$

Thus we see that, when expressed in terms of ξ and η the wave equation (3) simplifies dramatically to the form (6), and we say that (6) is the *canonical* form of (3). This equation is simple because it can be solved directly by integrating twice. First, integrating with respect to η gives

$$\frac{\partial u}{\partial \xi} = \int 0\, d\eta = f(\xi) \tag{7}$$

where the 'constant of integration' f is an arbitrary function of ξ. Next, the integration of (7) with respect to ξ gives

$$u(\xi,\eta) = \int f(\xi)\, d\xi + \phi(\eta) = \phi(\eta) + \psi(\xi) \tag{8}$$

where ϕ and ψ are arbitrary functions of one variable. Therefore, if u is to be a solution of wave equation it must be given by

$$u(x,t) = \phi(x+ct) + \psi(x-ct). \tag{9}$$

[1] This process of factoring the differential operator in the wave equation into first-order operators, thereby reducing the second-order wave equation to a pair of first-order equations, is also available for parabolic and elliptic equations, but the resulting system of first-order equations is not advantageous.

(It is worthwhile to note that by $\phi(x+ct)$ we mean $\phi(\eta)$ evaluated at $\eta = x+ct$ and $\psi(x-ct)$ is $\psi(\xi)$ evaluated at $\xi = x-ct$.) Since second-order derivative is appearing in the wave equation, the functions ϕ and ψ need to be twice differentiable. This is the d'Alembert's form of the general solution of wave equation (3). It is one of the few cases where the general solution of a partial differential equation can be found. However, finding the precise form of the arbitrary functions ϕ and ψ that satisfy given initial data is not always easy. The initial conditions must give just enough information to evaluate ϕ and ψ, which are functions of the single variables $\eta = x+ct$ and $\xi = x-ct$ respectively.

If we consider the time t as a parameter, the transformation $\xi = x-ct$ represents a translation of the coordinate system to the right by the amount ct. Since this translation is proportional to the time, a point $\xi =$ constant moves to the right with speed c. That is, a solution of the form

$$u(x,t) = \psi(x-ct)$$

represents a wave traveling with velocity c with its shape unchanged. Similarly, $u = \phi(x+ct)$ represents wave traveling to the left (velocity $-c$) with its shape unchanged.

It is clear from equation (9) that any solution of wave equation (3) is the sum of a wave traveling to the left with velocity $-c$ and one traveling to the right with velocity c. Since the two waves travel in opposite direction, the shape of $u(x,t)$ will in general changes with time.

Now we proceed to determine the form functions ϕ and ψ from the given initial conditions. Since the equation is second order in t, two initial conditions are necessary. Let the initial conditions are given as

$$u(x,0) = F(x), \qquad u_t(x,0) = G(x). \tag{10}$$

Since the general solution satisfies the initial condition $u(x,0) = F(x)$, we have

$$F(x) = \phi(x) + \psi(x). \tag{11}$$

That is, the functions ϕ and ψ should be so selected that their sum gives $F(x)$. Again, from chain rule and equation (8)

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = -c\frac{\partial u}{\partial \xi} + c\frac{\partial u}{\partial \eta} = -c\frac{d\psi}{d\xi} + c\frac{d\phi}{d\eta} \\ u_t &= -c\psi'(\xi) + c\phi'(\eta) \end{aligned}$$

Returning to original variables,

$$u_t(x,t) = c\phi'(x+ct) - c\psi'(x-ct).$$

Therefore, using the derivative initial condition $u_t(x,0) = G(x)$, we have

$$G(x) = c\phi'(x) - c\psi'(x) \tag{12}$$

Integrating this equation with respect to x to yield

$$\phi(x) - \psi(x) = \frac{1}{c}\int_{x_0}^{x} G(s)\,ds \tag{13}$$

where the constant of integration has been incorporated in the lower limit by introducing an arbitrary constant x_0. Solving equations (11) and (13) for ϕ and ψ, gives

$$\phi(x) = \frac{1}{2}F(x) + \frac{1}{2c}\int_{x_0}^{x} G(s)\,ds \tag{14a}$$

$$\psi(x) = \frac{1}{2}F(x) - \frac{1}{2c}\int_{x_0}^{x} G(s)\,ds \tag{14b}$$

Therefore the complete solution to the initial value problem is given by

$$\begin{aligned} u(x,t) &= \phi(x+ct) + \psi(x-ct) \\ &= \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} G(s)\,ds + \frac{1}{2}F(x-ct) + \frac{1}{2c}\int_{x-ct}^{x_0} G(s)\,ds \end{aligned} \tag{15}$$

where

$$\phi(x+ct) = \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} G(s)\,ds \tag{16a}$$

$$\psi(x-ct) = \frac{1}{2}F(x-ct) + \frac{1}{2c}\int_{x-ct}^{x_0} G(s)\,ds \tag{16b}$$

We can rewrite the solution (15) in the form

$$u(x,t) = \frac{1}{2}\left[F(x+ct) + F(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds. \tag{17}$$

This is known as the *d'Alembert's solution* to the wave equation. As mentioned earlier, it gives rise to waves propagating in the $+x$ and $-x$ directions with a speed of c.

The physical significance of the d'Alembert's solution can be explained by looking at the form of equation in (15). Clearly, $\psi(x-ct)$ represents a traveling wave propagating in the positive x-direction with speed c and its shape unchanged. Similarly, $\phi(x+ct)$ is also a traveling wave propagating in the negative x-direction with the same speed c without change of shape. A wave propagating in the positive x-direction with a constant speed c without change of shape is shown in Fig. 1.

The wave equation (3) has two sets of characteristic curves consisting of straight lines. The members of the set with slope $1/c$ and $-1/c$ are called right-running and left-running characteristics respectively. They are sketched in Fig. 2.

2.1 Domain of dependence and range of influence

It is interesting to know which part of the initial data (or in other words which part of the domain) affects the value of u at a given point $P(x_p,t_p)$. From d'Alembert's solution (17)

$$u(x_p,t_p) = \frac{1}{2}\left[F(x_p+ct_p) + F(x_p-ct_p)\right] + \frac{1}{2c}\int_{x_p-ct_p}^{x_p+ct_p} G(s)\,ds$$

we can see that the value of $u(x_p,t_p)$ depends on the initial data F at points x_p-ct_p and x_p+ct_p. The value of $u(x_p,t_p)$ also depends on values of G in the interval $[x_p-ct_p, x_p+ct_p]$ which is cut out of the initial line by the two characteristics $x-ct = x_p-ct_p$ and $x+ct = x_p+ct_p$ with slope $\pm(1/c)$ passing through the point $P(x_p,t_p)$. The interval $I = [x_p-ct_p, x_p+ct_p]$ is called the *domain of dependence* of

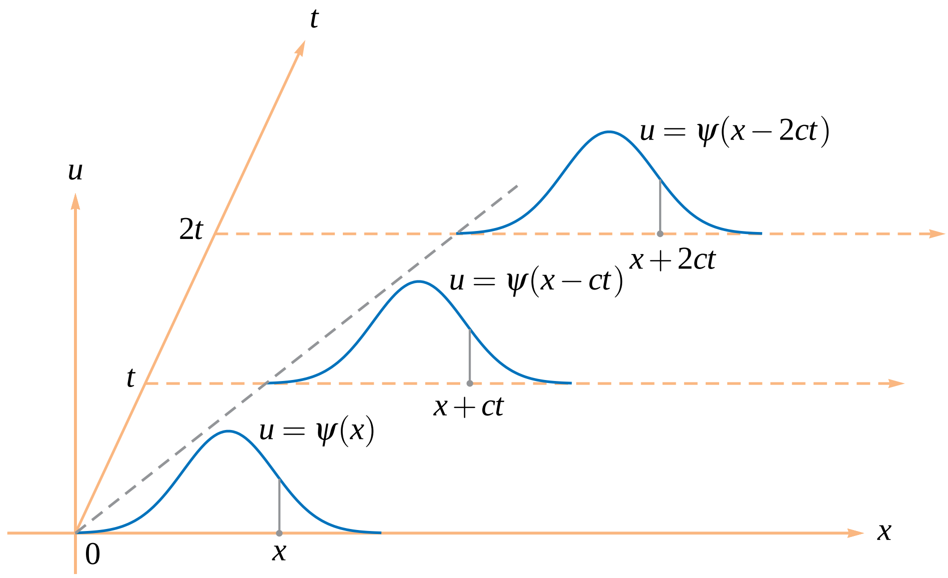

Figure 1: Traveling wave propagating in the positive x-direction.

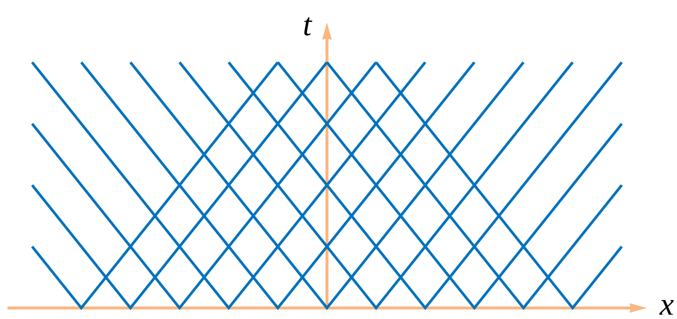

Figure 2: Characteristic lines of wave equation (3).

the solution at $P(x_p,t_p)$ as shown in Fig. 3. This behavior is to be expected because the effects of the initial data propagate at the finite speed c. Thus the only part of the initial data that can influence the solution at x_p at time t_p must be within ct_p units of x_p. This is precisely the data given in the interval $[x_p - ct_p, x_p + ct_p]$. The initial data outside this interval have no influence on the solution at $P(x_p,t_p)$.

Since the solution $u(x,t)$ at every point (x,t) inside the triangular region D in this figure is completely determined by the initial data on the interval $[x_p - ct_p, x_p + ct_p]$, the region D is called the *domain of determinacy* of the solution corresponding to initial data specified on the domain of dependence $[x_p - ct_p, x_p + ct_p]$.

Fig. 4 shows that the two characteristics that emerges into the upper half of the (x,t) plane from a point $P(a,0)$ on the initial data line as t increases. The initial data $F(x)$ at a point a will propagate at speed c whereas the effect of $G(x)$ propagates at all speeds up to c. This infinite sector R represents all points in the upper half of the (x,t) plane that can be influenced by the initial data at P and is called the *range of influence* of the point $P(a,0)$.

Now consider a general point $P(x_p,t_p)$ in the solution domain, $D(x,t)$ as shown in Fig. 5. The

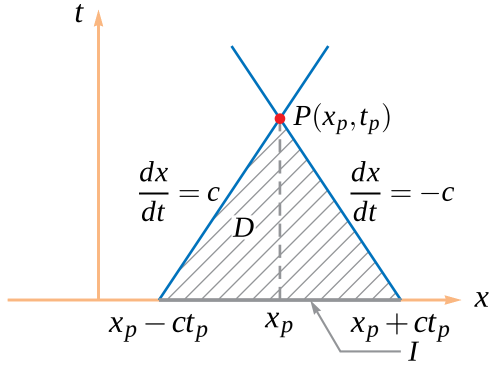

Figure 3: Domain of dependence I and region of determinacy D of point $P(x_p,t_p)$.

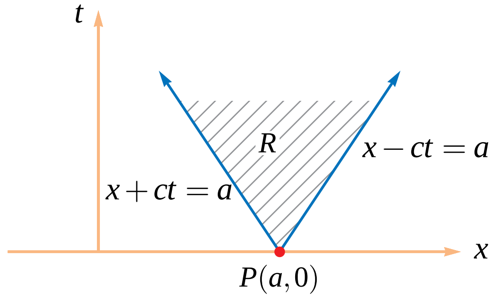

Figure 4: Range of influence R of point $P(a,0)$.

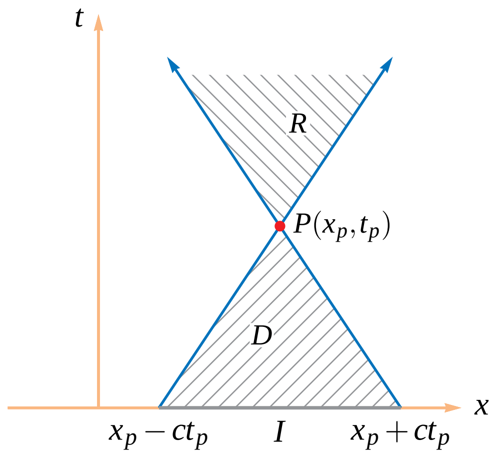

Figure 5: Domain of dependence I, region of determinacy D, and range of influence R of point P.

domain of dependence of point $P(x_p,t_p)$ is defined as the interval I on the initial data line upon which the solution at point $P(x_p,t_p)$, depends. The range of influence R of point $P(x_p,t_p)$ is defined as the region of the solution domain in which the solution is influenced by the solution at point $P(x_p,t_p)$. The triangular region D in Fig. 5, which is completely determined by the initial data on the interval I, is called the domain of determinacy of the solution corresponding to the domain of dependence I.

Sometimes the domain of determinacy plus the interval $[x_p - ct_p, x_p + ct_p]$ is referred to as the *domain of dependence* or *past history* of point $P(x_p,t_p)$.

2.2 d'Alembert's solution for the $G = 0$

The behaviour of the solution can be best understood by considering two special cases, viz., $G(x) = 0$ and $F(x) = 0$. Let us first consider the case where $G(x) = 0$.

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} && -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= F(x) && -\infty < x < \infty \\ u_t(x,0) &= 0 && -\infty < x < \infty \end{aligned} \tag{18}$$

In this case, the d'Alembert's solution (17) takes the form

$$u(x,t) = \frac{1}{2}\left[F(x+ct) + F(x-ct)\right] \tag{19}$$

To fix the idea, we consider $F(x) = Ae^{-x^2}$ (where A is a constant) as the initial wave (data) and is shown in Fig. 6 (top-left plot). The solution is given by

$$u(x,t) = \frac{A}{2}\left[e^{-(x+ct)^2} + e^{-(x-ct)^2}\right]$$

At time $t > 0$ we note that u is the average of $F(x+ct)$ and $F(x-ct)$, which is just the average of the two waves resulting from shifting $F(x)$ to the left ct units and to the right ct units. A sequence of time snapshots at $t_1 < t_2 < t_3 < t_4$ in Fig. 6 shows schematically how the two signals $F(x+ct)$ and $F(x-ct)$ are averaged to produce $u(x,t)$. The waves are initially superimposed and is equal to $F(x)$. It then splits into two waves moving in opposite directions each at speed c. The left moving wave travels along the characteristics $x+ct =$ constant and the right moving wave travels along the characteristics $x-ct =$ constant. Therefore there are two families of characteristic lines along which disturbances propagate.

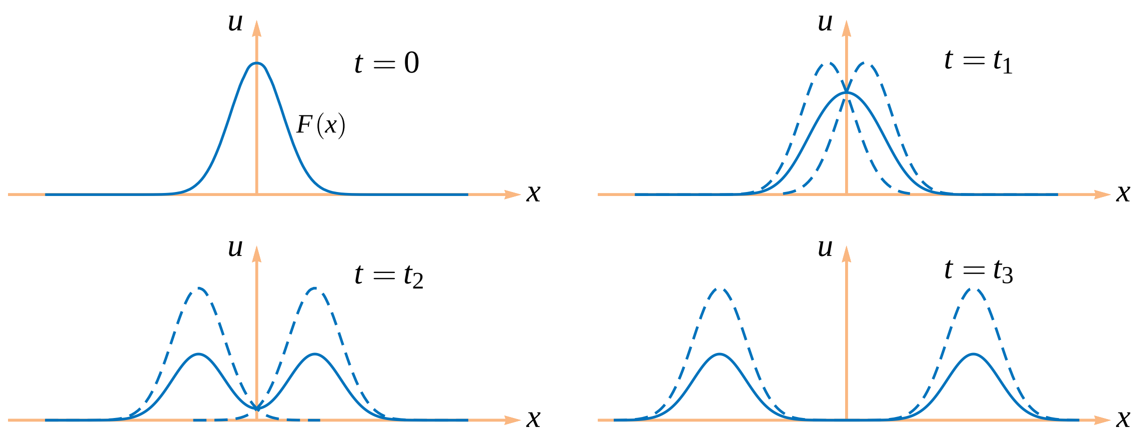

Figure 6: Time snapshots showing $u(x,t)$ (solid) as the average of the two profiles $F(x+ct)$ and $F(x-ct)$ (dashed).

Example 1

Solve the Cauchy problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} && -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= F(x) = A\sin x && -\infty < x < \infty \\ u_t(x,0) &= G(x) = 0 && -\infty < x < \infty \end{aligned}$$

The d'Alembert's solution (19) takes the form:

$$u(x,t) = \frac{A}{2}[\sin(x+ct) + \sin(x-ct)] = A\sin x \cos ct.$$

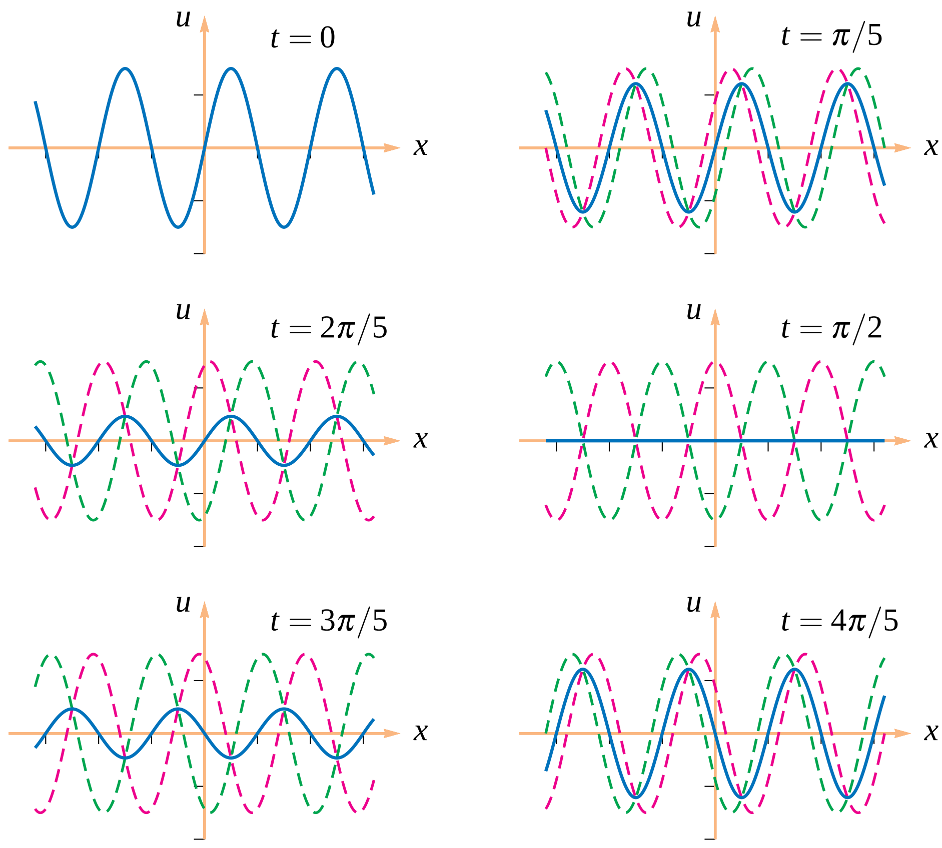

Figure 7: Time snapshots showing $u(x,t)$ (solid) as the average of the two profiles $F(x+ct)$ and $F(x-ct)$ (dashed).

Figure 7 shows snapshots of the solution for $c = 1$ and $A = 1$ at various representative times. It can be seen that the solution represents a standing wave with fixed nodes. It is the result of the product of a spatial shape ($\sin x$) and a temporal harmonic or oscillation ($\cos ct$). The shape is sometimes called the eigenfunction. The amplitude of the wave oscillates between -1 and 1.

Example 2

Solve the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty, \quad t > 0$$
$$u(x,0) = F(x) \qquad -\infty < x < \infty$$
$$u_t(x,0) = 0 \qquad -\infty < x < \infty$$

where

$$F(x) = A|\sin x|$$

The d'Alembert's solution (19) takes the form:

$$u(x,t) = \frac{A}{2}[|\sin(x+ct)| + |\sin(x-ct)|] = A|\sin x||\cos ct|.$$

Figure 8 shows snapshots of the solution for $c = 1$ and $A = 1$ at various representative times. The amplitude of the wave oscillates between 1 and 0.

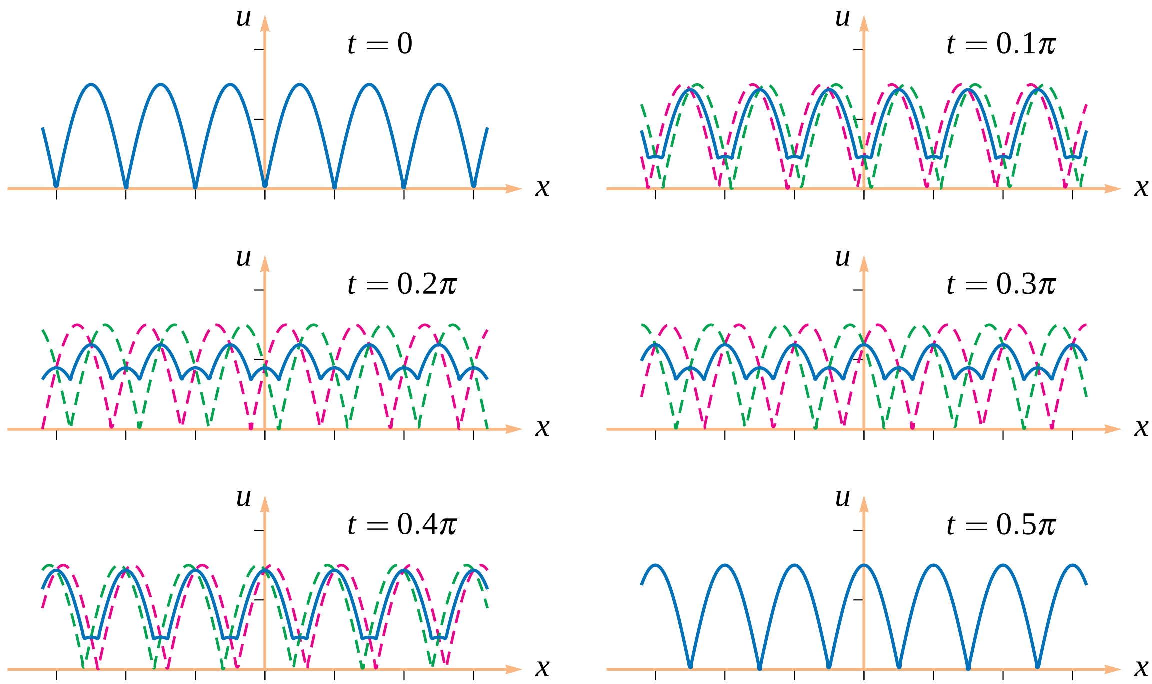

Figure 8: Time snapshots showing $u(x,t)$ (solid) as the average of the two profiles $F(x+ct)$ and $F(x-ct)$ (dashed).

Example 3

Solve the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = F(x) \qquad -\infty < x < \infty$$

$$u_t(x,0) = 0 \qquad -\infty < x < \infty$$

where

$$F(x) = \begin{cases} 0 & -\infty < x < -\pi \\ A\sin x & -\pi \le x \le \pi \\ 0 & \pi < x < \infty \end{cases}$$

which is only nonzero in the finite interval $-\pi \le x \le \pi$. Such localized initial conditions are said to have compact support, where the support of the data is the finite interval having nonzero data. The d'Alembert's solution is given by (19):

$$u(x,t) = \frac{1}{2}[F(x+ct) + F(x-ct)]$$

In order to account for the piecewise nature of $F(x)$ we can make use of the Heaviside step function

$$H(x-a) = \begin{cases} 0 & x < a \\ 1 & x > a \end{cases}$$

Thus, we have

$$F(x) = [H(x+\pi) - H(x-\pi)]A\sin x$$

and the solution

$$u(x,t) = \frac{A}{2}[H(x+ct+\pi) - H(x+ct-\pi)]\sin(x+ct) + \frac{A}{2}[H(x-ct+\pi) - H(x-ct-\pi)]\sin(x-ct)$$

For $c = 1$, the solution becomes

$$u(x,t) = \frac{A}{2}[H(x+t+\pi) - H(x+t-\pi)]\sin(x+t) + \frac{A}{2}[H(x-t+\pi) - H(x-t-\pi)]\sin(x-t)$$

Figure 9 shows snapshots of the solution for $A = 1$ at various representative times. It shows how the initial sinusoidal wave decompose into two opposite moving propagating waves, one of which moves to the left and the other two the right with unit speed $c = 1$. Before separating they interact linearly, but after separating at $t = \pi$, each propagating wave becomes replica of the initial wave profile $F(x)$, though with half of its amplitude.

Even though this solution is not twice differentiable, it can be shown to be a "weak" solution as discussed previously with regard to the solution of first-order wave equation.

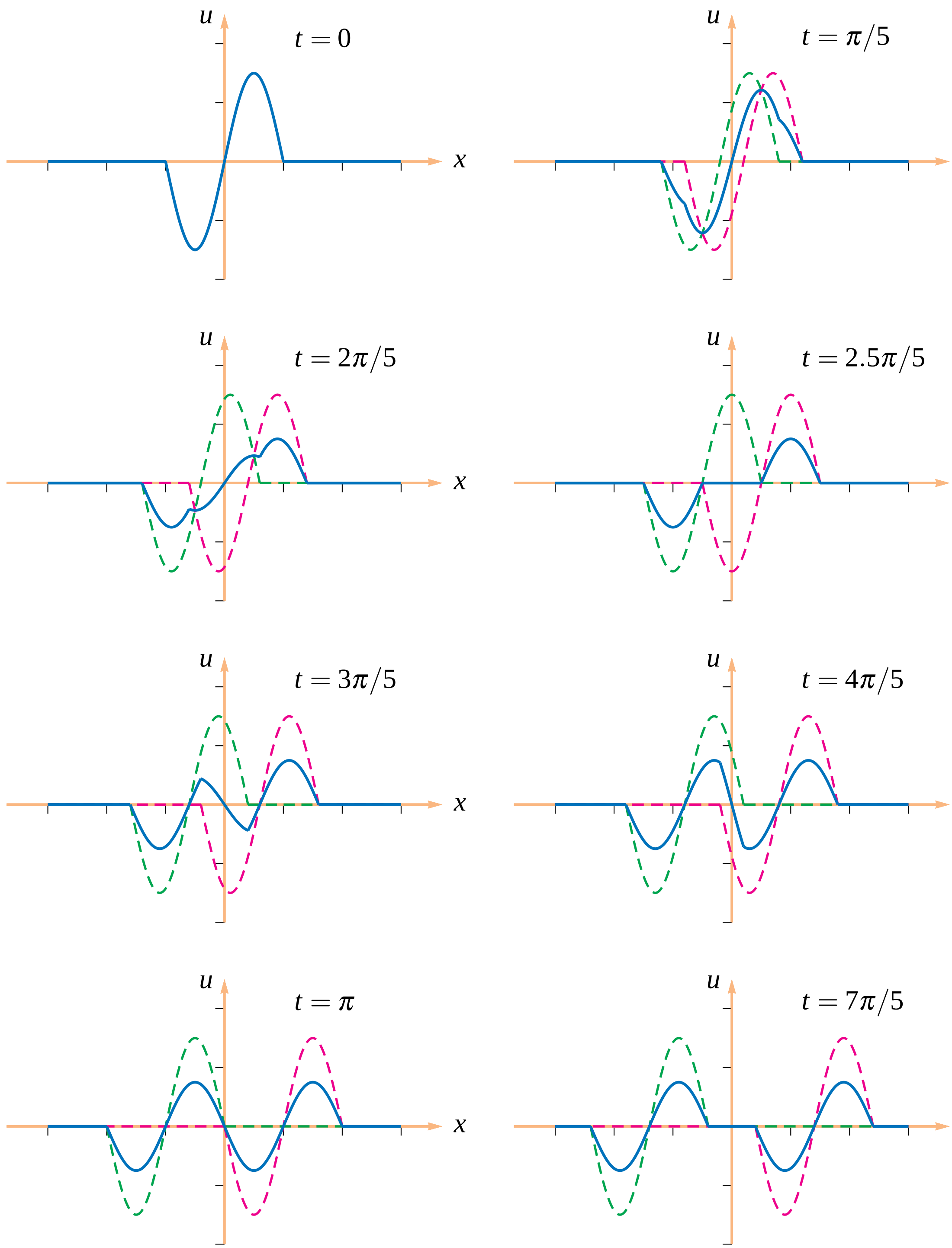

Figure 9: Time snapshots showing $u(x,t)$ (solid) as the average of the two profiles $F(x+ct)$ and $F(x-ct)$ (dashed).

Example 4

Solve the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = F(x) \qquad -\infty < x < \infty$$

$$u_t(x,0) = 0 \qquad -\infty < x < \infty$$

where

$$F(x) = Ae^{-x^2} \sin x$$

and A being a constant. For $c = 1$, the d'Alembert's solution is given by (19):

$$u(x,t) = \frac{A}{2}[e^{-(x+t)^2} \sin(x+t) + e^{-(x-t)^2} \sin(x-t)]$$

Figure 10 shows snapshots of the solution at various representative times. The initial waveform splits into two waves having half the initial amplitude and traveling in the positive and negative x directions with a speed of unity.

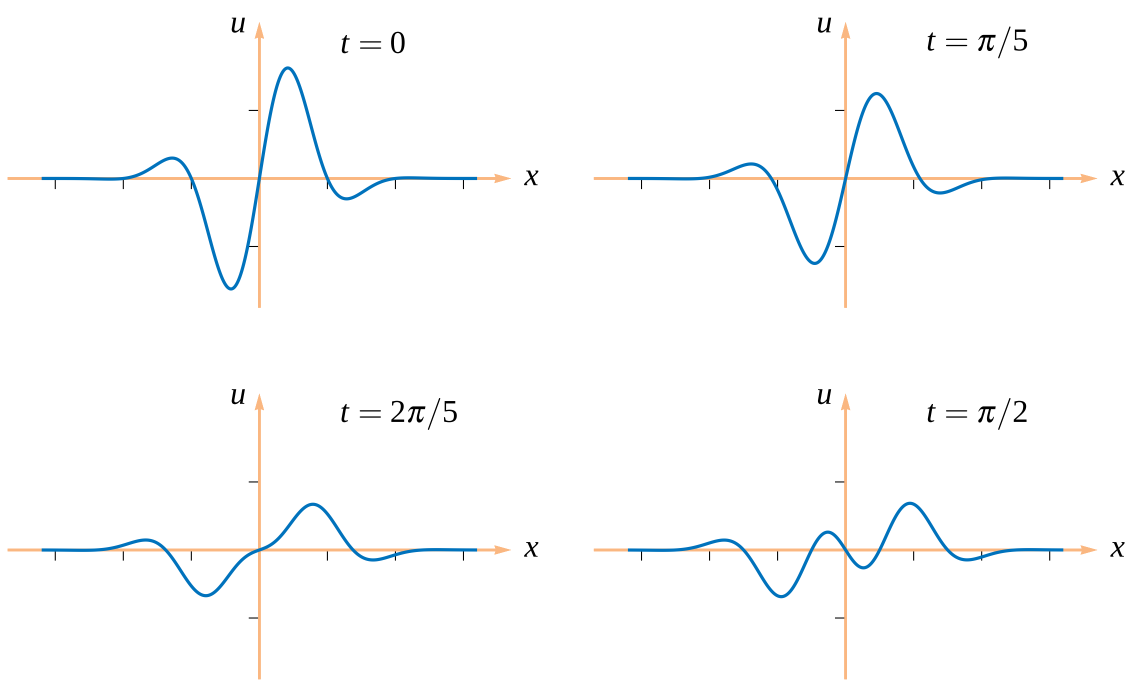

Figure 10: Time snapshots of $u(x,t)$.

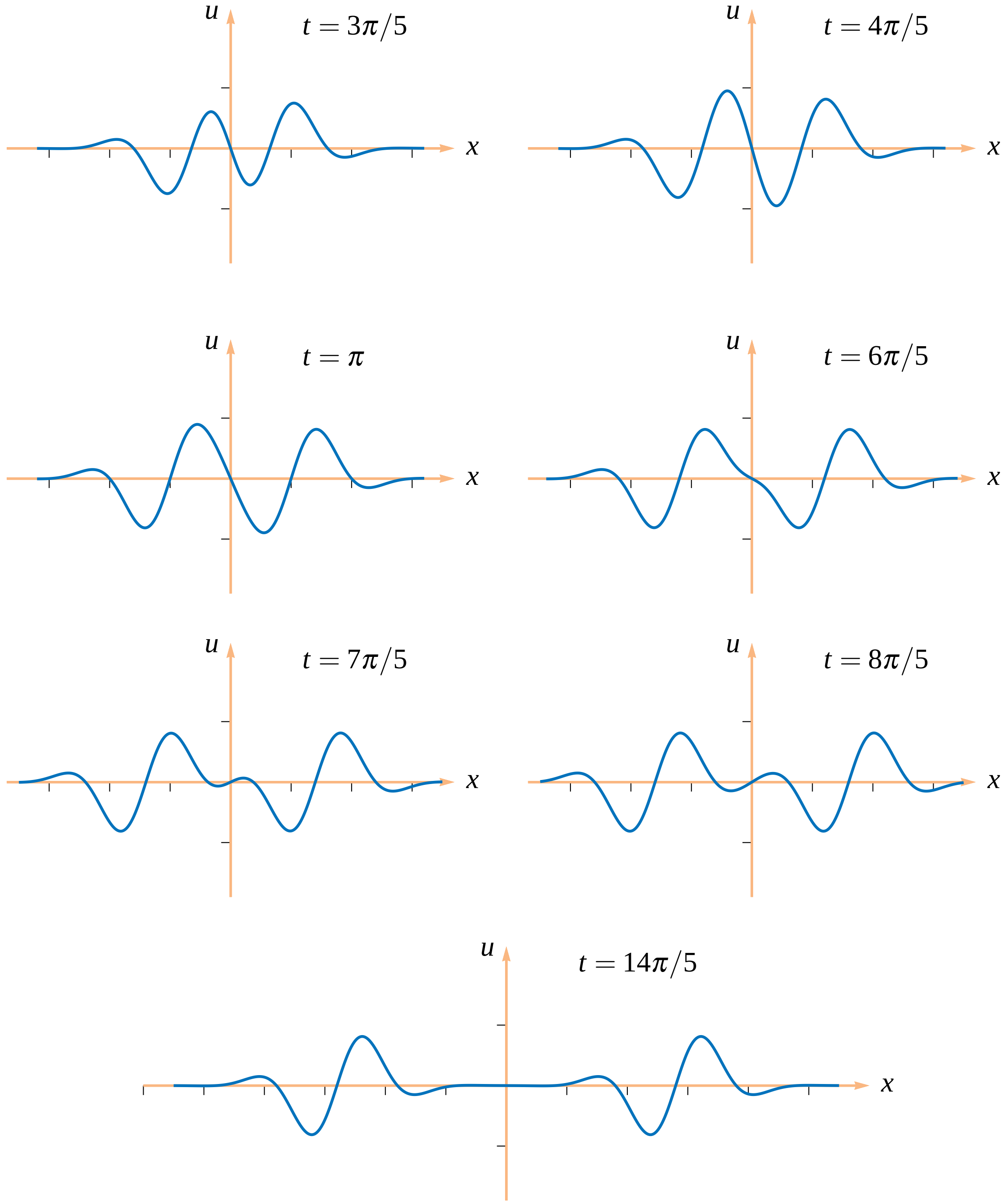

Figure 10 continued.

Example 5

Solve the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty, \quad t > 0$$
$$u(x,0) = F(x) \qquad -\infty < x < \infty$$
$$u_t(x,0) = 0 \qquad -\infty < x < \infty$$

where the initial waveform is given by the hat function

$$F(x) = \begin{cases} x+1 & -1 \le x \le 0 \\ 1-x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

This problem corresponds to a vibrating infinitely long string due to "three-finger pluck", with all three fingers removed from the string at once.

The initial waveform can be represented by making use of Heaviside step function as follows:

$$F(x) = [H(x+1) - H(x)](x+1) + [H(x) - H(x-1)](1-x)$$

The d'Alembert's solution is then given by (19):

$$u(x,t) = \frac{1}{2}[F(x+ct) + F(x-ct)]$$

Thus, we have

$$\begin{aligned} u(x,t) = &\frac{1}{2}[H(x+ct+1) - H(x+ct)](x+ct+1) + [H(x+ct) - H(x+ct-1)](1-x-ct) + \\ &\frac{1}{2}[H(x-ct+1) - H(x-ct)](x-ct+1) + [H(x-ct) - H(x-ct-1)](1-x+ct) \end{aligned}$$

Figure 11 shows snapshots of the solution at various representative times. It shows how the initial triangular wave decompose into two opposite moving propagating waves, one of which moves to the left and the other two the right with unit speed $c = 1$ and amplitude half of the original wave.

Example 6

Solve the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty, \quad t > 0$$
$$u(x,0) = F(x) \qquad -\infty < x < \infty$$
$$u_t(x,0) = 0 \qquad -\infty < x < \infty$$

where the initial waveform is given by the hat function

$$F(x) = \begin{cases} 0 & -\infty < x < -1 \\ 1 & -1 \le x \le 1 \\ 0 & 1 < x < \infty \end{cases}$$

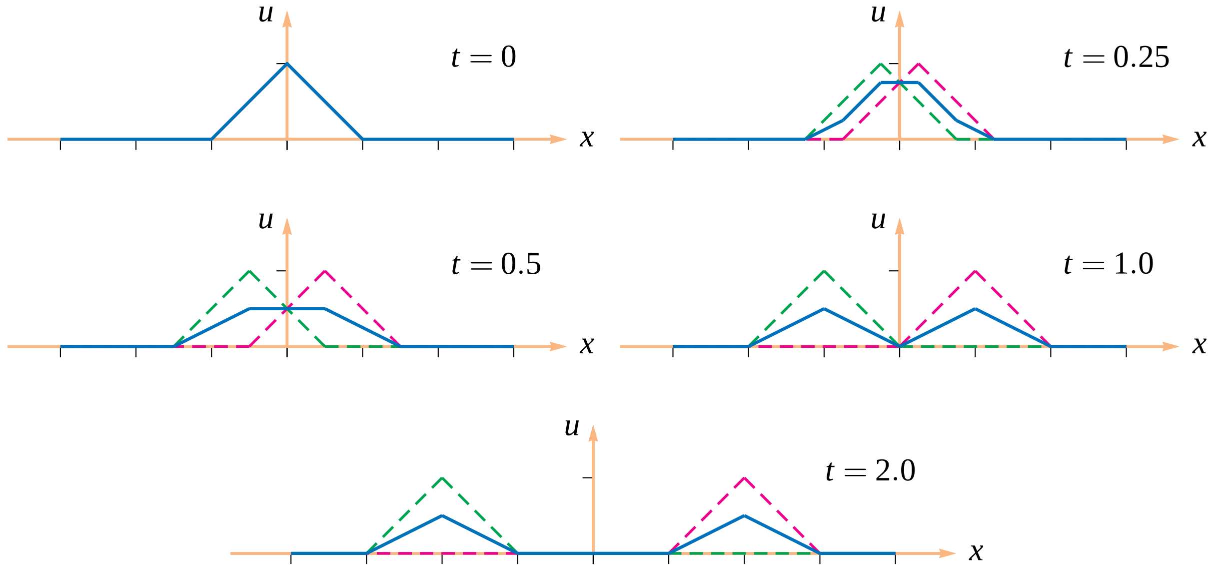

Figure 11: Time snapshots showing $u(x,t)$ (solid) as the average of the two profiles $F(x+ct)$ and $F(x-ct)$ (dashed).

This problem corresponds to a vibrating infinitely long string due to "two-finger pluck", with two fingers removed from the string at once.

The initial waveform can be represented by making use of Heaviside step function as follows:

$$F(x) = H(1+x) + H(1-x) - 1 = H(x+1) - H(x-1)$$

The d'Alembert's solution is then given by (19):

$$u(x,t) = \frac{1}{2}[F(x+ct) + F(x-ct)]$$

Thus, we have

$$u(x,t) = \frac{1}{2}[H(x+ct+1) - H(x+ct-1] + \frac{1}{2}[H(x-ct+1) - H(x-ct-1]$$

Figure 12 shows snapshots of the solution at various representative times. It shows how the initial triangular wave decompose into two opposite moving propagating waves, one of which moves to the left and the other two the right with unit speed $c = 1$ and amplitude half of the original wave.

2.3 d'Alembert's solution for the $F = 0$

Having discussed some of the d'Alembert's solution for the case $G = 0$, let us now consider the case where $F(x) = 0$. We consider the initial value problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} && -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= 0 && -\infty < x < \infty \\ u_t(x,0) &= G(x) && -\infty < x < \infty \end{aligned} \tag{20}$$

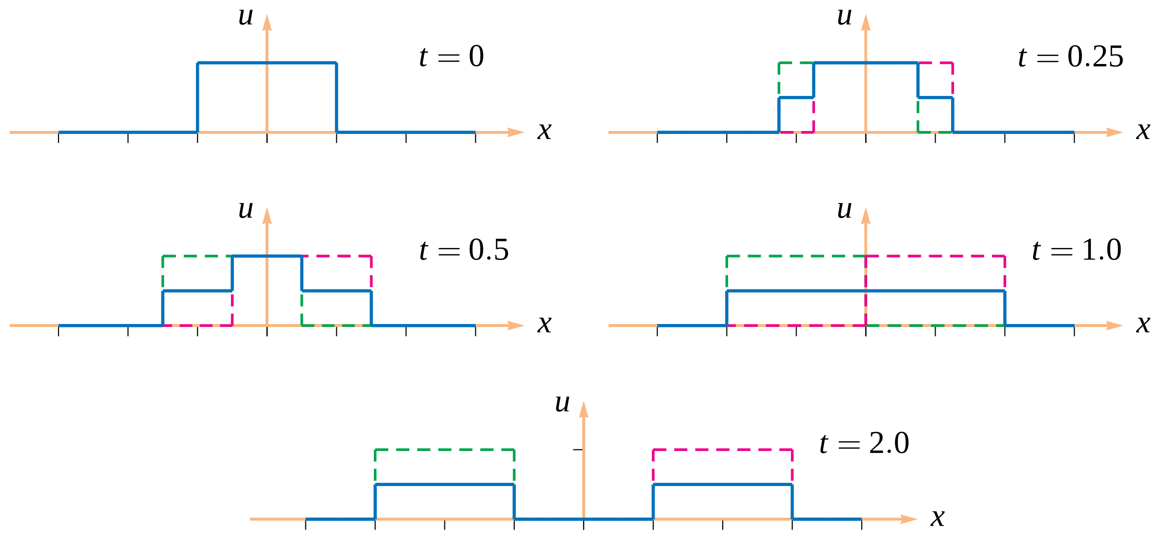

Figure 12: Time snapshots showing $u(x,t)$ (solid) as the average of the two profiles $F(x+ct)$ and $F(x-ct)$ (dashed).

For this case the d'Alembert's solution assumes the form

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds = \frac{1}{2}[g(x+ct) - g(x-ct)] \tag{21}$$

where

$$g(x) = \frac{1}{c}\int_{x_0}^{x} G(s)\,ds.$$

Fundamental Solution

Consider an infinite string with zero initial displacement and initial velocity given by an impulse function at $x_0 = 0$. That is, we have from (20)

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \qquad -\infty < x < \infty, \quad t > 0 \tag{22a}$$

$$F(x) = 0 \tag{22b}$$

$$G(x) = \delta(x) \tag{22c}$$

where δ is the Dirac delta function

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0, \\ 0 & \text{if } x \neq 0 \end{cases}$$

with the property $\int_{-\infty}^{\infty} \delta(x)dx = 1$. We calculate $g(x)$ as follows

$$g(x) = \frac{1}{c}\int_{-\infty}^{x} \delta(s)\,ds = \frac{1}{c} \times \begin{cases} 0 & -\infty < x < 0 \\ 1 & 0 > x > \infty \end{cases}$$
$$= \frac{1}{c}H(x)$$

where H is the Heaviside function. Therefore, the solution (21) takes the form

$$u(x,t) = \frac{1}{2}[g(x+ct) - g(x-ct)] = \frac{1}{2c}[H(x+ct) - H(x-ct)] \tag{23}$$

This is called the *Fundamental Solution* of wave equation. It is worthwhile to note that the problem posed in (22) is equivalent to the following nonhomogeneous problem

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = \delta(x)\delta(t) \qquad -\infty < x < \infty, \quad t > 0 \tag{24a}$$

$$F(x) = 0 \tag{24b}$$

$$G(x) = 0 \tag{24c}$$

Here the source term acts at location $x = 0$ and switch-on time $t = 0$. The fundamental solution for an arbitrary source location $x = \xi$ and switch-on time $t = \tau$ is obtained from (23) by translation. Thus, the solution of

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = \delta(x-\xi)\delta(t-\tau) \qquad -\infty < x < \infty, \quad t > 0 \tag{25a}$$

$$F(x) = 0 \tag{25b}$$

$$G(x) = 0 \tag{25c}$$

is

$$u(x,t) = \frac{1}{2c}[H(x-\xi+c(t-\tau)) - H(x-\xi-c(t-\tau))] \tag{26}$$

The fundamental solution (23) represents a uniform front of height $u = 1/2c$ propagating with constant speed c in the positive and negative x directions, as shown in Fig. 13. If we subdivide the xt-plane into the three regions as indicated, we note that in region A, $x+ct < 0$, $x-ct < 0$. Therefore, $H(x+ct) = 0$, $H(x-ct) = 0$, and hence $u = 0$. In region R, $x+ct > 0$, $x-ct < 0$. Therefore, $H(x+ct) = 1$, $H(x-ct) = 0$, and hence $u = 1/2c$. Finally, in region B, $x+t > 0$, $x-t > 0$. Therefore, $H(x+t) = H(x-t) = 1$, and $u = 0$. The triangular domain R is called the range of influence of the source at $(0,0)$.

In contrast with the diffusion equation discussed in the previous chapter, wave equation has a distinct "disturbance wave front", which propagates with a speed of c and separates regions of disturbed and undisturbed "motion". It is interesting to note that as t approaches infinity, the string will reach a state of rest, but it will not assume its original position. This displacement is known as the residual displacement.

Example 7

Consider an infinite string with zero initial displacement and initial velocity given by the square wave function (hammer blow). That is, we have $F(x) = 0$ and

$$G(x) = \begin{cases} 0 & -\infty < x < -1 \\ 1 & -1 \leq x \leq 1 \\ 0 & 1 < x < \infty \end{cases}$$

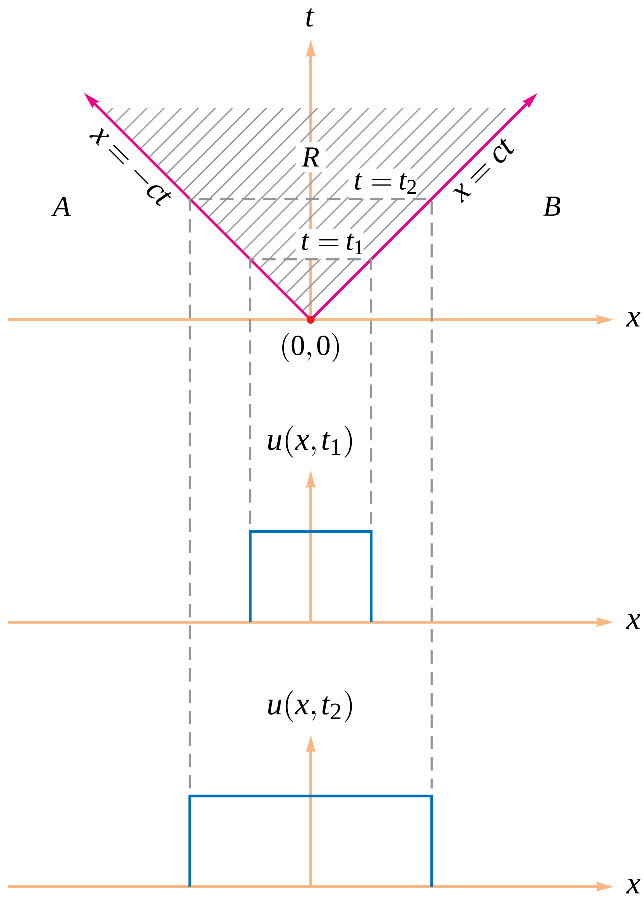

Figure 13: Range of influence of point source at $(0,0)$.

which be represented by making using Heaviside step function as follows:

$$G(x) = H(x+1) - H(x-1)$$

The initial square wave function is shown in figure 14.

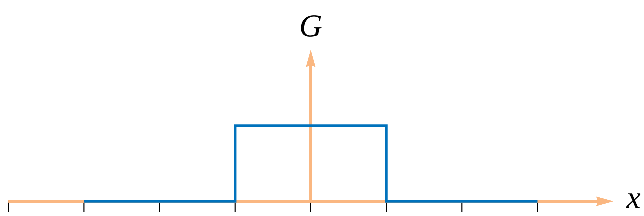

Figure 14: Initial velocity profile, $G(x)$.

We now calculate $g(x)$:

$$g(x) = \frac{1}{c}\int_{-\infty}^{x} G(s)\,ds = \frac{1}{c} \times \begin{cases} \int_{-\infty}^{x} 0\,ds = 0 & -\infty < x < -1 \\ \int_{-\infty}^{-1} 0\,ds + \int_{-1}^{x} 1\,ds = x+1 & -1 \leq x \leq 1 \\ \int_{-\infty}^{-1} 0\,ds + \int_{-1}^{1} 1\,ds + \int_{1}^{x} 0\,ds = 2 & 1 < x < \infty \end{cases}$$

$$= \begin{cases} 0 & -\infty < x < -1 \\ (x+1)/c & -1 \leq x \leq 1 \\ 2/c & 1 < x < \infty \end{cases}$$

$$= \frac{1}{c}[H(x+1) - H(x-1)](x+1) + \frac{2}{c}H(x-1)$$

$$= \frac{1}{c}[(x+1)H(x+1) - (x-1)H(x-1)]$$

Therefore, the solution of (20) takes the form

$$\begin{aligned} u(x,t) &= \frac{1}{2}[g(x+ct) - g(x-ct)] \\ &= \frac{1}{2c}[(x+ct+1)H(x+ct+1) - (x+ct-1)H(x+ct-1)] \\ &\qquad - \frac{1}{2c}[(x-ct+1)H(x-ct+1) - (x-ct-1)H(x-ct-1)] \end{aligned}$$

For $c = 1$, the solution becomes

$$\begin{aligned} u(x,t) &= \frac{1}{2}[(x+t+1)H(x+t+1) - (x+t-1)H(x+t-1)] \\ &\qquad - \frac{1}{2}[(x-t+1)H(x-t+1) - (x-t-1)H(x-t-1)] \end{aligned}$$

As in the previous case, the effect of an initial velocity G is a wave spreading out at speed c in opposite directions on the x-axis. After some time t the two functions $g(x)/2$ and $-g(x)/2$ move a distance ct. Thus, the graph of u at time t is obtained by summing the ordinates of the displaced graphs as shown in Figure 15. The string will reach a state of rest as t tends to infinity with a residual displacement, $u_{\text{st}} = 1$.

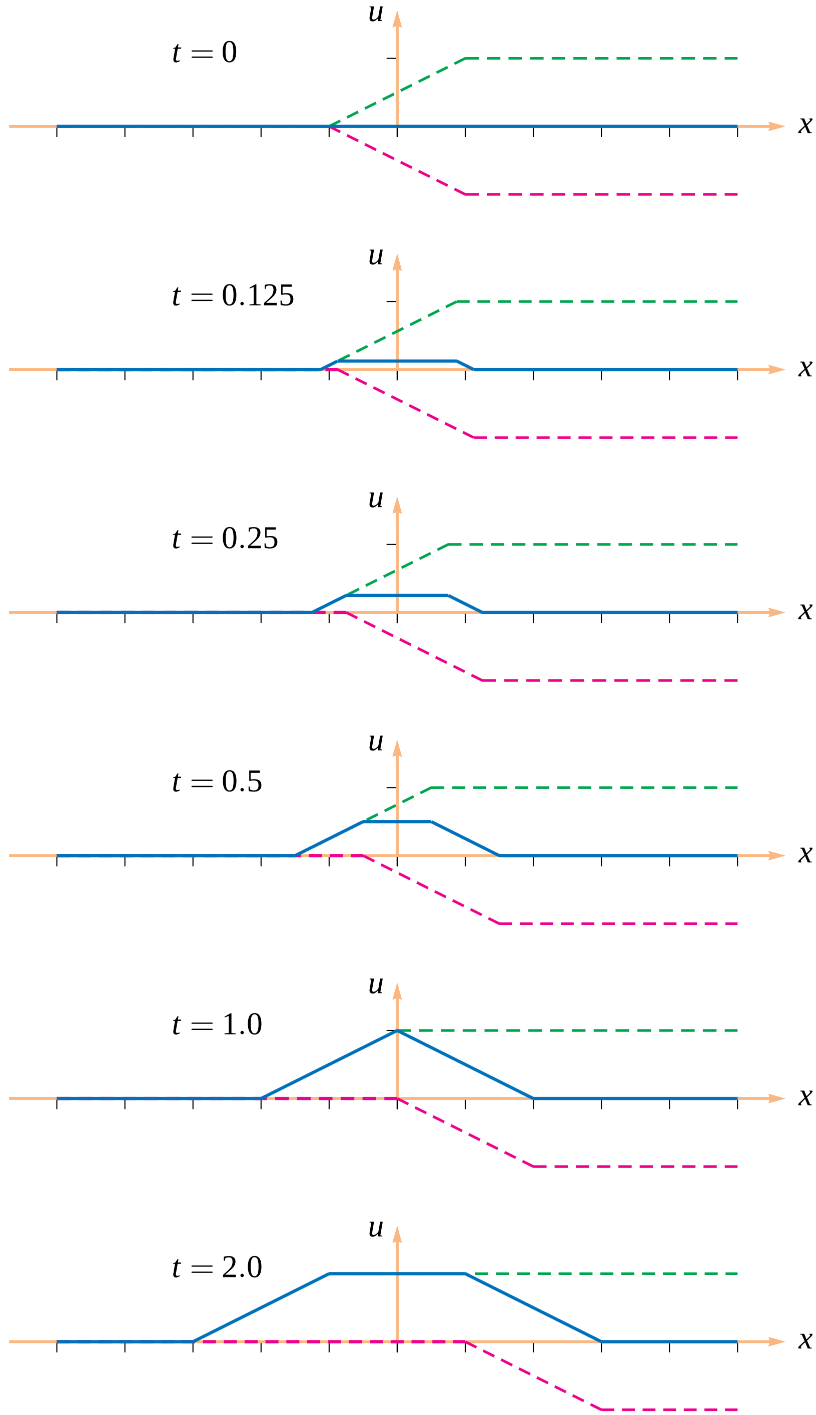

Figure 15: Time snapshots showing $u(x,t)$ (solid) as the average of the two profiles $g(x+ct)$ and $-g(x-ct)$ (dashed).

Example 8

Consider a an infinite string with zero initial displacement ($F(x) = 0$) and initial velocity

$$G(x) = \frac{1}{a} e^{-|x|/a}$$

A sketch of $G(x)$ for $a = 0.7$ is shown in figure 16.

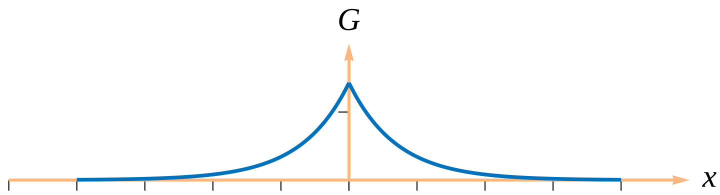

Figure 16: Initial velocity profile, $G(x)$.

We first calculate $g(x)$, $g(x+ct)$, and $g(x-ct)$ as follows:

$$g(x) = \frac{1}{c}\int_{-\infty}^{x} G(s)\,ds = \frac{1}{ac} \times \begin{cases} \int_{-\infty}^{x} e^{x/a}\,ds = ae^{x/a} & -\infty < x < 0 \\ \int_{-\infty}^{0} e^{x/a}\,ds + \int_{0}^{x} e^{-x/a}\,ds = 2a - ae^{-x/a} & 0 < x < \infty \end{cases}$$

$$= \begin{cases} \frac{1}{c} e^{x/a} & -\infty < x < 0 \\ \frac{1}{c}\left(2 - e^{-x/a}\right) & 0 < x < \infty \end{cases}$$

$$g(x+ct) = \begin{cases} \frac{1}{c} e^{(x+ct)/a} & -\infty < x < -ct \\ \frac{1}{c}\left(2 - e^{-(x+ct)/a}\right) & -ct < x < \infty \end{cases}$$

$$g(x-ct) = \begin{cases} \frac{1}{c} e^{(x-ct)/a} & -\infty < x < ct \\ \frac{1}{c}\left(2 - e^{-(x-ct)/a}\right) & ct < x < \infty \end{cases}$$

Therefore, the solution of (20) takes the form

$$u(x,t) = \frac{1}{2}[g(x+ct) - g(x-ct)] = \begin{cases} \frac{1}{2c}\left(e^{(x+ct)/a} - e^{(x-ct)/a}\right) & -\infty < x < -ct \\ \frac{1}{2c}\left(2 - e^{-(x+ct)/a} - e^{(x-ct)/a}\right) & -ct < x < ct \\ \frac{1}{2c}\left(e^{-(x-ct)/a} - e^{-(x+ct)/a}\right) & ct < x < -\infty \end{cases}$$

For $c = 1$, the solution becomes

$$u(x,t) = \begin{cases} \frac{1}{2}\left(e^{(x+t)/a} - e^{(x-t)/a}\right) & -\infty < x < -t \\ \frac{1}{2}\left(2 - e^{-(x+t)/a} - e^{(x-t)/a}\right) & -t < x < t \\ \frac{1}{2}\left(e^{-(x-t)/a} - e^{-(x+t)/a}\right) & t < x < -\infty \end{cases}$$

For $c = 1$, the limit of the displacement $u(x,t)$ at any fixed x, as $t \to \infty$ is found to be 1.

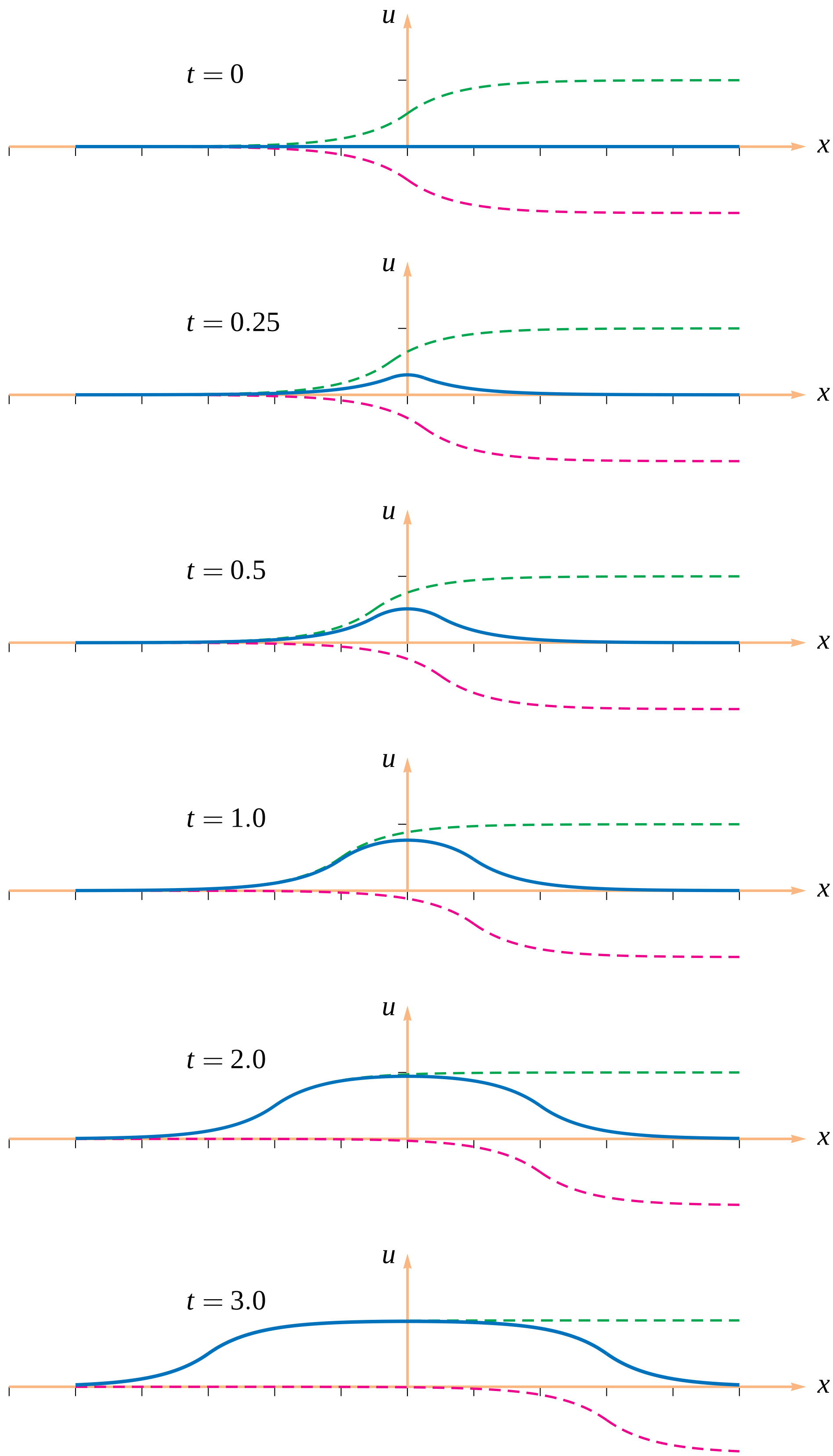

Figure 17: Time snapshots showing $u(x,t)$ (solid) as the average of the two profiles $g(x+ct)$ and $-g(x-ct)$ (dashed).

Example 9

Solve the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = F(x) \qquad -\infty < x < \infty$$

$$u_t(x,0) = G(x) \qquad -\infty < x < \infty$$

where

$$F(x) = 0 \qquad \text{and} \qquad G(x) = A\sin x$$

We calculate

$$\int_{x-ct}^{x+ct} G(s)\,ds = -A[\cos(x+ct) - \cos(x-ct)]$$

Therefore, the solution becomes

$$u(x,t) = -\frac{A}{2c}[\cos(x+ct) - \cos(x-ct)] = A\sin x \sin ct$$

Figure 18 shows snapshots of the solution for $c = 1$ and $A = 1$ at various representative times. It can be seen that the solution represents a standing wave with fixed nodes. The amplitude of the wave oscillates between 1 and 0. This is the same type of motion we obtained for the case of sinusoidal displacement, $F(x) = A\sin x$ with no initial velocity. However, there is a phase difference of $\pi/2$ is observed in comparison to the other case.

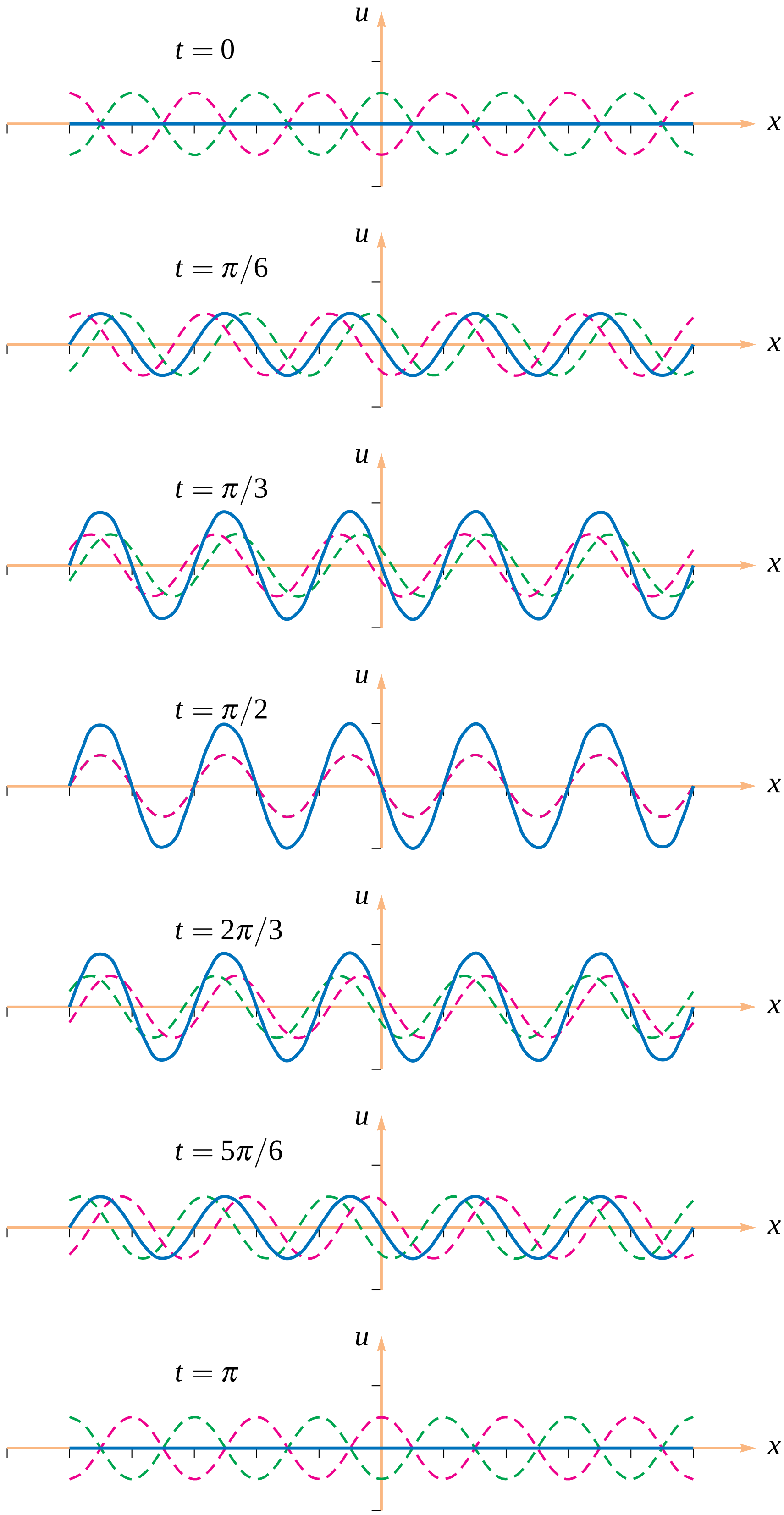

Figure 18: Time snapshots showing $u(x,t)$ (solid) as the average of the two profiles $g(x+ct)$ and $-g(x-ct)$ (dashed).