Delta Function and Heaviside Function

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We discuss some of the basic properties of the generalized functions, viz., Dirac-delta function and Heaviside step function.

Heaviside step function

The one-dimensional Heaviside step function centered at a is defined in the following way

$$H(x-a) = \begin{cases} 0 & \text{if } x < a, \\ 1 & \text{if } x > a. \end{cases} \tag{1a}$$

For $a = 0$ the discontinuity is at $x = 0$, thus we have

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases} \tag{1b}$$

The heaviside function is displayed in Fig. 1.

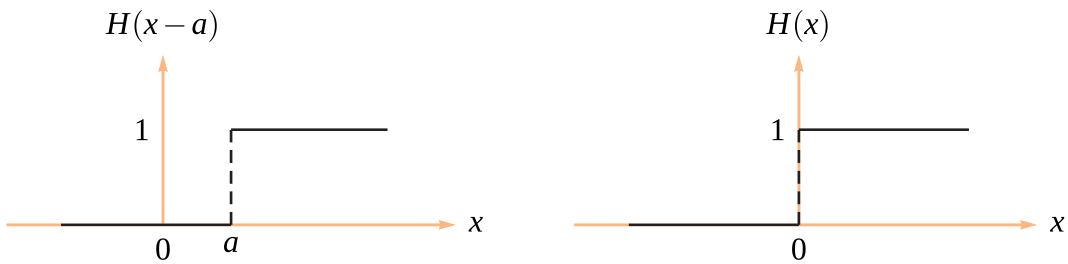

Figure 1: The Heaviside functions $H(x-a)$ and $H(x)$.

Dirac-delta function

To understand the behaviour of Dirac-delta function (or delta function, for short) $\delta(x)$, we consider the rectangular pulse function

$$\Delta(x,a) = \begin{cases} h & \text{if} \quad a - \dfrac{1}{2h} < x < a + \dfrac{1}{2h}, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

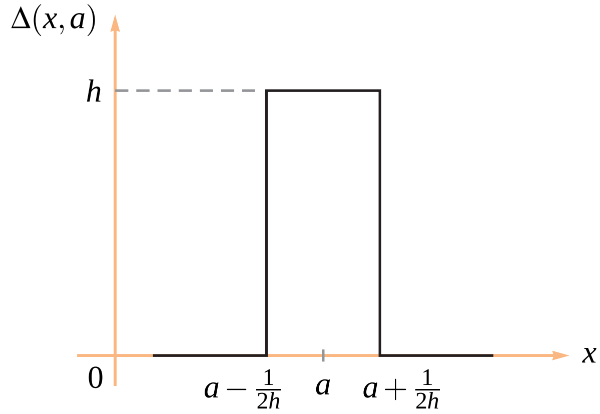

Figure 2: The pulse function.

From figure 2, it can be seen that as $h \to \infty$, the amplitude of pulse becomes very large and its width becomes very small so that for any value of h, the integral of the rectangular pulse

$$\int_{\alpha}^{\beta} \Delta(x,a)\,dx = 1$$

if the the integral of definition $(a - \frac{1}{2h}, a + \frac{1}{2h})$ lies in the interval (α, β), and zero if range of integration does not contain the pulse. Now, we can define the Dirac-delta function $\delta(x-a)$ located at the point $x = a$ as

$$\delta(x-a) = \lim_{h\to\infty} \Delta(x,a) = \lim_{h\to\infty} \Delta(x-a). \tag{3}$$

To understand the significance of $\delta(x-a)$, let us consider the integral

$$\int_{\alpha}^{\beta} f(x)\Delta(x,a)\,dx$$

where $f(x)$ is an arbitrary continuous function defined over $\alpha < x < \beta$. From mean value theorem, we have

$$\int_{\alpha}^{\beta} f(x)\Delta(x-a)\,dx = \int_{a-\frac{1}{2h}}^{a+\frac{1}{2h}} f(x)\Delta(x,a)\,dx = \left[\left(a+\frac{1}{2h}\right) - \left(a-\frac{1}{2h}\right)\right] f(\xi)D(\xi) = \frac{1}{h}hf(\xi) = f(\xi)$$

where ξ is an unknown point within the interval $(a - \frac{1}{2h}, a + \frac{1}{2h})$. As $h \to \infty$, we have $\Delta(x - a) \to \delta(x-a)$, and the point ξ in the interval $(a - \frac{1}{2h}, a + \frac{1}{2h})$ moves closer to a, and hence $f(\xi) \to f(a)$. Thus we have the fundamental property of the delta function

$$\int_{\alpha}^{\beta} f(x)\delta(x-a)\,dx = \begin{cases} f(a) & \text{if } \alpha < a < \beta, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

For example,

$$\int_{1}^{6} (3x-1)\delta(x-2)\,dx = 5.$$

This shows the filtering property of the delta function when it occurs under the integral sign, because from all the values of $f(x)$ in the interval of integration, delta function $\delta(x-a)$ has selected the value $f(a)$ at the location where it is acting. Delta functions are not ordinary functions in the sense that we can ask for the value of $\delta(x-a)$ at say $x=7$. They are examples of what are called "generalized functions", and they are characterized by their effect on other functions through integral (4).

If $f(x)=1$, we obtain the following relation

$$\int_{\alpha}^{\beta} \delta(x-a)\, dx = \begin{cases} 1 & \text{if } \alpha < x < \beta, \\ 0 & \text{otherwise.} \end{cases} \tag{5a}$$

where the limit of the integration can be extended from $-\infty$ to ∞. Thus, we have

$$\int_{-\infty}^{\infty} \delta(x-a)\, dx = 1. \tag{5b}$$

The fact that the delta function is not an ordinary function and thus cannot be represented on a graph is clearly apparent from definition (2), because

$$\delta(x-a) = \begin{cases} \infty & \text{if } x = a, \\ 0 & \text{if } x \neq a. \end{cases} \tag{6}$$

such that

$$\int_{-\infty}^{\infty} \delta(x-a)\, dx = 1.$$

If we replace the upper limit of the integral ∞, by a finite value x, then we have the following property

$$\int_{-\infty}^{x} \delta(x-a)\, dx = \begin{cases} 0 & \text{if } x < a, \\ 1 & \text{if } x > a. \end{cases} \tag{7}$$

Comparing equations (1a) and (7) we get the following relation between Heaviside function and delta function

$$H(x-a) = \int_{-\infty}^{x} \delta(x-a)\, dx. \tag{8}$$

Differentiation of equation (8) with respect to x, yields the following relation

$$\frac{dH(x-a)}{dx} = \delta(x-a). \tag{9}$$

If the delta function is acting at the origin, i.e., if $a=0$, we have the fundamental property of the delta function

$$\int_{\alpha}^{\beta} f(x)\delta(x)\, dx = \begin{cases} f(0) & \text{if } \alpha < 0 < \beta, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

and if $f(x)=1$ in the above equation, we have

$$\int_{\alpha}^{\beta} \delta(x)\, dx = \begin{cases} 1 & \text{if } \alpha < 0 < \beta, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x)\, dx = 1. \tag{11}$$

The delta function can then be defined as

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases} \tag{12}$$

and the relationship between Heaviside function and delta function is given by

$$\frac{dH(x)}{dx} = \delta(x) \tag{13}$$

and

$$H(x) = \int_{-\infty}^{x} \delta(x)\,dx = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases} \tag{14}$$

Regularized Dirac-delta function

Instead of using the limit of ever-narrowing rectangular pulse of unit area when defining delta function, any similar functions can be used, provided their integral is unity and their amplitude increase as their pulse-like property narrows. For example, a regularized (smeared-out) delta function in an interval $(a-\varepsilon, a+\varepsilon)$ is given by

$$\delta_\varepsilon(x-a) = \begin{cases} \dfrac{1}{2\varepsilon}\left[1+\cos\left(\frac{\pi(x-a)}{\varepsilon}\right)\right] & \text{if} \quad a-\varepsilon < x < a+\varepsilon, \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

where ε is a parameter that determines the size of the width of smearing. The variation of $\delta_\varepsilon(x)$ with x for different values of ε is shown in figure. Note that the function value of the peak (which is at the point $x = a$) is $1/\varepsilon$.

The property given by equation (5) is also valid for regularized delta function. To show this, we integrate $\delta_\varepsilon(x)$ over the interval $[a-\varepsilon, a+\varepsilon]$;

$$\begin{aligned} \int_{a-\varepsilon}^{a+\varepsilon} \delta_\varepsilon(x)dx &= \int_{a-\varepsilon}^{a+\varepsilon} \frac{1}{2\varepsilon}\left[1+\cos\left(\frac{\pi(x-a)}{\varepsilon}\right)\right] dx \\ &= \int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon}\left[1+\cos\left(\frac{\pi y}{\varepsilon}\right)\right] dy \qquad \text{(putting } y = x-a\text{)} \\ &= \frac{1}{2\varepsilon}\left[y + \frac{\sin\left(\frac{\pi y}{\varepsilon}\right)}{\pi/\varepsilon}\right]_{-\varepsilon}^{\varepsilon} \\ &= \frac{1}{2\varepsilon}[(\varepsilon+0)-(-\varepsilon+0)] \\ &= 1. \end{aligned}$$

A useful property of the regularized delta function is given by

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} f(x)\delta_\varepsilon(x-a)dx = f(a). \tag{16}$$

If the delta function is acting at the origin, i.e., if $a = 0$, the regularized delta function defined by (15) becomes

$$\delta_\varepsilon(x) = \begin{cases} \dfrac{1}{2\varepsilon}\left[1+\cos\left(\frac{\pi x}{\varepsilon}\right)\right] & \text{if} \quad -\varepsilon < x < \varepsilon, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Another example of regularized delta function is a sequence of bell-shaped pulses defined as

$$\delta_k(x-a) = \frac{1}{k\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-a}{k}\right)^2} \tag{18}$$

where k is a parameter. This regularized delta function approaches to delta function $\delta(x-a)$ as $k \to 0$. That is,

$$\delta(x-a) = \lim_{k\to 0} \frac{1}{k\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-a}{k}\right)^2}. \tag{19}$$

Note that the integral of $\delta_k(x-a)$, i.e.,

$$\int_{-\infty}^{\infty} \frac{1}{k\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-a}{k}\right)^2} = 1$$

for all values of $k > 0$, and the bell-shaped pulses defined in this way becomes narrower as $k \to 0$ as displayed in Fig. 3. If the delta function is acting at the origin, i.e., if $a = 0$, the regularized delta function defined by (18) becomes

$$\delta_k(x) = \frac{1}{k\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{k}\right)^2}. \tag{20}$$

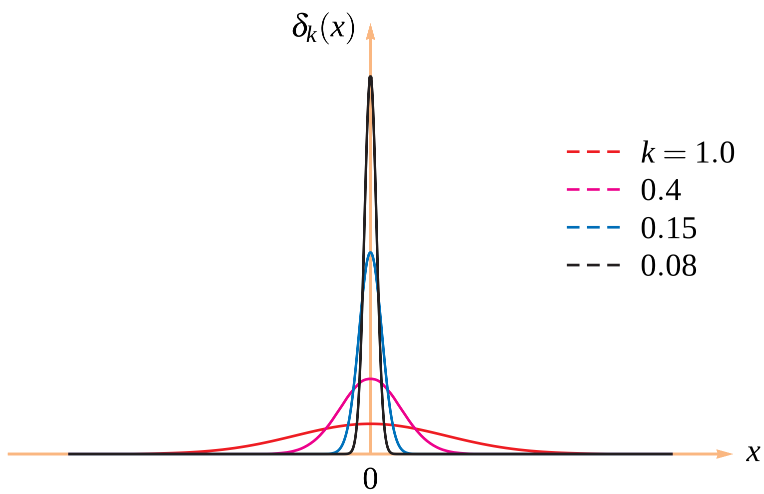

Figure 3: The regularized delta function as defined in (20).