Work-Energy Theorem and Energy Conservation

A. Salih
Department of Aerospace Engineering
Indian Institute of Space Science and Technology, Thiruvananthapuram
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Work-Energy Theorem

The kinetic energy of a particle of mass m, moving with a speed v, is defined as

$$T = \frac{1}{2}mv^2. \tag{1}$$

Let us consider a particle that moves from point 1 to point 2 under the action of a force $\overline{F}$. The total work done on the particle by the force as the particle moves from 1 to 2 is, by definition, the line integral

$$W_{12} = \int_1^2 \overline{F} \cdot d\overline{s} \tag{2}$$

where $d\overline{s} = \overline{v}\,dt$ is the displacement vector along the particle's trajectory. If the particle undergoes an infinitesimal displacement $d\overline{s}$ under the action of force $\overline{F}$, the scalar product

$$dW = \overline{F} \cdot d\overline{s} \tag{3}$$

is the infinitesimal work done by the force $\overline{F}$ as the particle undergoes the displacement $d\overline{s}$ along the particle's trajectory. We use the Newton's second law of motion

$$\overline{F} = \frac{d(m\overline{v})}{dt}$$

in the equation (3) to obtain an expression for the infinitesimal work

$$dW = \frac{d(m\overline{v})}{dt} \cdot \overline{v}\,dt = \frac{d}{dt}\left(\frac{1}{2}m\overline{v} \cdot \overline{v}\right) dt = d\left(\frac{1}{2}mv^2\right).$$

Since the scalar quantity $\frac{1}{2}mv^2$ is the kinetic energy of the particle, it follows that

$$dW = dT. \tag{4}$$

Equation (4) is the differential form of the work-energy theorem: It states that the differential work of the resultant of forces acting on a particle is equal, at any time, to the differential change in the kinetic energy of the particle. Integrating equation (4) between point 1 and point 2, corresponding to the velocities v_1 and v_2 of the particle, we get

$$W_{12} = \int_1^2 dW = \int_1^2 dT = T_2 - T_1 = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2. \tag{5}$$

This is the work-energy theorem, which states that *the work done by the resultant force* $\overline{F}$ *acting on a particle as it move from point 1 to point 2 along its trajectory is equal to the change in the kinetic energy* $(T_2 - T_1)$ *of the particle during the given displacement.* When the body is accelerated by the resultant force, the work done on the body can be considered a transfer of energy to the body, where it is stored as kinetic energy.

Energy Conservation Theorem

If there exists a scalar function $\phi(x,y,z,t)$, so that we could write

$$\overline{F} = \nabla\phi \tag{6}$$

we shall say that the vector field $\overline{F}$ is a potential field. The scalar function $\phi(x,y,z,t)$ is then called the *potential function* of the field. The vector field $\overline{F}$ is called conservative if ϕ does not explicitly depend on time. The potential function $\phi(x,y,z)$, in this case, is called the *force potential*.

It is easy to show that if the force field is conservative the work done in moving the particle from 1 to 2 is independent of the path connecting 1 and 2. From equation (2), the total work done on the particle by the force $\overline{F}$ as it moves from 1 to 2 is given by

$$W_{12} = \int_1^2 \overline{F} \cdot d\overline{s}.$$

Then, for a conservative force field we have

$$W_{12} = \int_1^2 \overline{F} \cdot d\overline{s} = \int_1^2 \nabla\phi \cdot d\overline{s} = \int_1^2 \frac{d\phi}{ds}\,ds = \int_1^2 d\phi = \phi_2 - \phi_1. \tag{7}$$

Thus, the total work done is equal to the difference in force potential no matter how the particle moves from 1 to 2. We also have the following differential relation

$$dW = \overline{F} \cdot d\overline{s} = d\phi. \tag{8}$$

If we now write $\phi(x,y,z) = -U(x,y,z)$ (inserting a minus sign for reasons of convention) and express the force as

$$\overline{F} = -\nabla U \tag{9}$$

then the scalar function U is called the *potential energy* of the particle. When $\overline{F}$ is expressed as in the above equation, the work done becomes

$$W_{12} = U_1 - U_2. \tag{10}$$

That is, the total work done is equal to the difference in potential energy $(U_1 - U_2)$ no matter how the particle moves from 1 to 2.

It may be noted that the line integral of the field $\overline{F} = -\nabla U$ along a closed curve (called circulation) is zero as shown below:

$$\oint_C \overline{F} \cdot d\overline{s} = -\oint_C dU = 0.$$

Comparing equations (5) and (10), it can be concluded that $T_1 + U_1 = T_2 + U_2$. It says that the quantity $T + U$ remains a constant as the particle moves from point 1 to point 2. Since 1 and 2 are arbitrary points, we have obtained the statement of conservation of total mechanical energy

$$E = T + U = \text{constant}. \tag{11}$$

Thus, the energy conservation theorem states that *the total energy of a particle in a conservative force field is constant.*

It is instructive to note that equation (6) does not uniquely determine the function ϕ. We could as well define $\overline{F} = \nabla\phi + c$, where c is any constant. Hence, the choice for the zero level of ϕ, and consequently U, is arbitrary.

We can verify directly from equation (11) that the total energy in a conservative field is *a constant* of the motion. We have

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dU}{dt}.$$

The kinetic energy term can be written as

$$\frac{dT}{dt} = \frac{1}{2}m\frac{dv^2}{dt} = m\frac{d\overline{v}}{dt} \cdot \overline{v} = \overline{F} \cdot \overline{v}.$$

The potential energy U depends on time only through the changing position of the particle: $U = U(\overline{s}(t)) = U(x(t), y(t), z(t))$. Thus, we have

$$\frac{dU}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt} = \nabla U \cdot \overline{v} = -\overline{F} \cdot \overline{v}.$$

It follows that

$$\frac{dE}{dt} = \overline{F} \cdot \overline{v} - \overline{F} \cdot \overline{v} = 0.$$

Thus, the total energy of the particle moving in a conservative force field is a constant during the motion.

Force-potential energy relation

Let us consider a conservative force

$$\overline{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}.$$

We then have

$$\overline{F} = \nabla\phi = -\nabla U.$$

Therefore, we have the following relations:

$$F_x = \frac{\partial \phi}{\partial x} = -\frac{\partial U}{\partial x}, \qquad F_y = \frac{\partial \phi}{\partial y} = -\frac{\partial U}{\partial y}, \qquad F_z = \frac{\partial \phi}{\partial z} = -\frac{\partial U}{\partial z}. \tag{12}$$

This shows that the partial derivative of force potential in a given direction gives the force in that direction. An example of a force that derives from a potential is gravitational force

$$\overline{F}_g = -\nabla U$$

which leads to the following equations

$$mg_x = -\frac{\partial U}{\partial x}, \qquad mg_y = -\frac{\partial U}{\partial y}, \qquad mg_z = -\frac{\partial U}{\partial z}, \tag{13}$$

where the gravitational acceleration vector $\overline{g} = (g_x, g_y, g_z)$. It follows that the negative of partial derivative of potential energy in a given direction gives the gravitational force in that direction.

If gravitational acceleration vector is given by

$$\overline{g} = g(0, 0, -1)$$

then we have

$$0 = -\frac{\partial U}{\partial x} \qquad 0 = -\frac{\partial U}{\partial y} \qquad -mg = -\frac{\partial U}{\partial z} \tag{14}$$

Integrating the last of the above equation to obtain

$$U = mgz + f(x,y).$$

Setting $f(x,y) = 0$, the potential energy of the particle in a gravitational field is given by

$$U = mgz$$

where $\overline{g}$ acts in the negative z direction. The total mechanical energy E is conserved when a particle moves under the action of the gravitational field.

Non-conservative force

An example of a force that does not derive from a potential is the frictional force $\overline{F}_{\text{fr}} = -k\overline{v}$, where k is the coefficient of friction. This force acts in the direction opposite to the particle's motion and is responsible for the drag force. The frictional force cannot be expressed as the gradient of a scalar function. This implies that in the presence of a frictional force, the total mechanical energy of a particle E is not conserved. The reason is that the friction causes the mechanical energy E to transform into heat. Energy conservation as a whole, of course, applies, i.e., the amount by which E decreases matches the amount of heat dissipated into the environment.

It is instructive to note that the work-energy theorem given by equation (5) is always true, whether or not the force $\overline{F}$ derives from a potential.