Kelvin–Helmholtz Instability

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One of the most well known instabilities in fluid mechanics is the instability at the interface between two horizontal parallel streams of different velocities and densities, with the heavier fluid at the bottom. This is called Kelvin–Helmholtz instability. The name is also commonly used to describe the instability where the variations of velocity and density are continuous. If the discontinuity exists only in the velocity (densities of both layer are same) such a configuration is called a *vortex sheet*.

Development of perturbation equations

Let us investigate the motion of the surface of discontinuity located at $y = 0$ in the unperturbed state (Fig. 1). Assume that the layers have infinite depth and that the interface has zero thickness. Let U_1 and ρ_1 be the velocity and density of the *basic state* in the upper layer of (x, y)-plane and U_2 and ρ_2 be those in the bottom layer.

Let us consider the equations that govern the flow, including any perturbation. The flow above the vortex sheet has a velocity potential ϕ_1 and that below the sheet ϕ_2. Incompressible, irrotational (potential) flows satisfy

$$\nabla^2\phi_1 = 0 \qquad \text{and} \qquad \nabla^2\phi_2 = 0 \tag{1}$$

with boundary conditions

$$\begin{aligned} \nabla\phi_1 &= U_1 \qquad \text{as} \quad y \to \infty \\ \nabla\phi_2 &= U_2 \qquad \text{as} \quad y \to -\infty \end{aligned} \tag{2}$$

These conditions require that the perturbation die out far from the interface. Suppose that due to a perturbation the interface is deformed and is described by the equation, $y = \eta(x, t)$. The perturbed interface can also be represented by a parametric equation of the form $f(x, y, t) = 0$. Since, $y - \eta(x, t) = 0$ at every point (x, y) on the interface at any time t, it is clear that

$$f(x, y, t) = y - \eta(x, t) \tag{3}$$

If the interface moves with a velocity $\overline{V}_{\text{int}}$, the kinematic condition at the interface is given by

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \overline{V}_{\text{int}} \cdot \nabla f = 0 \tag{4}$$

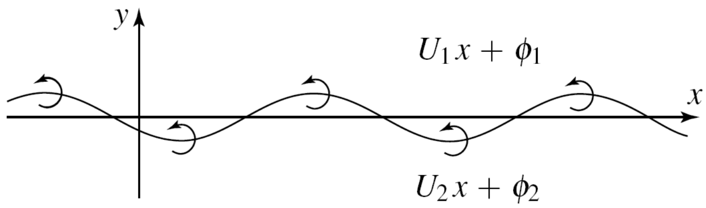

Figure 1: Kelvin–Helmholtz instability: Initial perturbed state

Using (3) equation (4) can be written as

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = v_\eta \qquad \text{at} \quad y = \eta(x,t) \tag{5}$$

where v_η is the vertical component of fluid velocity at the interface. Equation (5) is the *kinematic boundary condition* which states that the interface moves up and down with a velocity equal to the vertical component of the fluid velocity.

Considering particles just above the interface, the kinematic boundary condition requires

$$\frac{\partial \eta}{\partial t} + (U_1 + u_1')\frac{\partial \eta}{\partial x} = v_1' \qquad \text{at} \quad y = \eta(x,t)$$

where u_1' and v_1' are the components of the perturbed velocity in the x and y-directions respectively. Since the base flow is in the x-direction, we have $v_1' = \partial\phi_1/\partial y$. Therefore, the above equation may be rearranged as

$$\frac{\partial \phi_1}{\partial y} = \frac{\partial \eta}{\partial t} + (U_1 + u_1')\frac{\partial \eta}{\partial x} \qquad \text{at} \quad y = \eta(x,t) \tag{6}$$

A similar argument can be made for the lower layer, with the result that

$$\frac{\partial \phi_2}{\partial y} = \frac{\partial \eta}{\partial t} + (U_2 + u_2')\frac{\partial \eta}{\partial x} \qquad \text{at} \quad y = \eta(x,t) \tag{7}$$

This provides a boundary condition on ϕ_1 and ϕ_2.

The dynamic boundary condition is derived from the unsteady Bernoulli equation:

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{(\nabla\phi)^2}{2} + gy = C(t) \tag{8}$$

Since (8) holds on both sides of the surface, we have

$$\begin{aligned}\frac{\partial \phi_1}{\partial t} + \frac{p_1}{\rho_1} + \frac{(\nabla\phi_1)^2}{2} + gy = C_1 \\ \frac{\partial \phi_2}{\partial t} + \frac{p_2}{\rho_2} + \frac{(\nabla\phi_2)^2}{2} + gy = C_2\end{aligned} \tag{9}$$

The condition at the interface requires that pressure must be continuous across the interface (if surface tension is neglected). That is $p_1 = p_2$ at the interface and hence for $y = \eta$, we have

$$\rho_1 \left(\frac{\partial \phi_1}{\partial t} + \frac{(\nabla \phi_1)^2}{2} - C_1 \right) = \rho_2 \left(\frac{\partial \phi_2}{\partial t} + \frac{(\nabla \phi_2)^2}{2} - C_2 \right) \tag{10}$$

Equations (1), (2), (6), (7), and (10) govern the inviscid motion of a slip interface and flows on either side.

The basic flow U_1, U_2 satisfy the problem with $\eta = 0$, the dynamic boundary condition then reduces to

$$\rho_1 \left(\tfrac{1}{2} U_1^2 - C_1 \right) = \rho_2 \left(\tfrac{1}{2} U_2^2 - C_2 \right) \tag{11}$$

The flow is decomposed into a basic state plus perturbations. Thus, the potentials are

$$\begin{aligned} \phi_1 &= U_1 x + \phi_1' \\ \phi_2 &= U_2 x + \phi_2' \end{aligned} \tag{12}$$

where the first terms on the right-hand side represent the basic flow of uniform streams. When these relations are substituted into equation (1) we find that

$$\nabla^2 \phi_1' = 0 \qquad \text{and} \qquad \nabla^2 \phi_2' = 0 \tag{13}$$

While equation (2) shows that the perturbations die out at infinity:

$$\begin{aligned} \nabla \phi_1' &= 0 \qquad \text{as} \quad y \to \infty \\ \nabla \phi_2' &= 0 \qquad \text{as} \quad y \to -\infty \end{aligned} \tag{14}$$

The surface conditions (6) and (7) can be linearized by applying it at $y = 0$ instead of at $y = \eta$ and by dropping quadratic terms.

$$\begin{aligned} \frac{\partial \phi_1'}{\partial y} &= \frac{\partial \eta}{\partial t} + U_1 \frac{\partial \eta}{\partial x} \qquad \text{at} \quad y = 0 \\ \frac{\partial \phi_2'}{\partial y} &= \frac{\partial \eta}{\partial t} + U_2 \frac{\partial \eta}{\partial x} \qquad \text{at} \quad y = 0 \end{aligned} \tag{15}$$

Introducing the decomposition (12) into the dynamic boundary conditions (9), and requiring $p_1 = p_2$ at $y = \eta$, we perform following exercise to obtain a condition at the interface:

The first equation in (9) becomes

$$\frac{\partial}{\partial t} \left(U_1 x + \phi_1' \right) + \frac{p_1}{\rho_1} + \tfrac{1}{2} \left[\nabla \left(U_1 x + \phi_1' \right) \right]^2 + g\eta = C_1$$

$$\frac{\partial \phi_1'}{\partial t} + \frac{p_1}{\rho_1} + \frac{1}{2} \left[\left(\frac{\partial}{\partial x} \left(U_1 x + \phi_1' \right) \right)^2 + \left(\frac{\partial}{\partial y} \left(U_1 x + \phi_1' \right) \right)^2 \right] + g\eta = C_1$$

$$\frac{\partial \phi_1'}{\partial t} + \frac{p_1}{\rho_1} + \frac{1}{2} \left[\left(U_1 + \frac{\partial \phi_1'}{\partial x} \right)^2 + \left(\frac{\partial \phi_1'}{\partial y} \right)^2 \right] + g\eta = C_1$$

Linearize the above equation by neglecting the products of primed quantities, we get

$$\frac{\partial \phi_1'}{\partial t} + \frac{p_1}{\rho_1} + \frac{1}{2}\left(U_1^2 + 2U_1 \frac{\partial \phi_1'}{\partial x}\right) + g\eta = C_1$$

Solving for pressure

$$-p_1 = \rho_1 \left(\frac{\partial \phi_1'}{\partial t} + \tfrac{1}{2}U_1^2 + U_1 \frac{\partial \phi_1'}{\partial x} + g\eta - C_1\right)$$

A similar transformation can be done for the second equation in (9) to obtain

$$-p_2 = \rho_2 \left(\frac{\partial \phi_2'}{\partial t} + \tfrac{1}{2}U_2^2 + U_2 \frac{\partial \phi_2'}{\partial x} + g\eta - C_2\right)$$

Since $p_1 = p_2$ at $y = \eta$, the above two equations can be combined and the steady-flow relation (11) is substituted to obtain

$$\rho_1 \left(\frac{\partial \phi_1'}{\partial t} + U_1 \frac{\partial \phi_1'}{\partial x} + g\eta\right) = \rho_2 \left(\frac{\partial \phi_2'}{\partial t} + U_2 \frac{\partial \phi_2'}{\partial x} + g\eta\right) \qquad \text{at} \quad y = 0 \tag{16}$$

The mathematical problem for η, ϕ_1', and ϕ_2' consists of equations (13) to (15).

Normal-mode analysis

One can solve the three linear equations (15) (two equations) and (16) and determine the three functions η, ϕ_1', and ϕ_2', from which the stability of the system is found. All coefficients of the three equations are constants, and hence we can carry out the normal-mode analysis.

The flow has been divided into a steady basic flow and a time-dependent perturbation. The perturbation can be represented by a composition of the following normal-modes ansatz:

$$\eta = \hat{\eta} e^{ik(x-ct)} \tag{17}$$

$$\phi_1' = \hat{\phi}_1(y) e^{ik(x-ct)} \qquad \phi_2' = \hat{\phi}_2(y) e^{ik(x-ct)} \tag{18}$$

where k is real (and can be taken positive without any loss of generality), $c = c_r + ic_i$ is a complex wave speed. Note that $\hat{\eta}$ is the original amplitude of the interface displacement and is a constant. It specifies the size of all perturbed quantities. When $c_i > 0$, this displacement is unstable and grows exponentially in time.

Substitution of normal modes (18) into Laplace equations (13) would allow us determine the forms of the amplitude functions $\hat{\phi}_1$ and $\hat{\phi}_2$. For example, $\nabla^2 \phi_1' = 0$ gives

$$\nabla^2 \left(\hat{\phi}_1 e^{ik(x-ct)}\right) = 0$$

$$\frac{\partial^2}{\partial x^2}\left(\hat{\phi}_1 e^{ik(x-ct)}\right) + \frac{\partial^2}{\partial y^2}\left(\hat{\phi}_1 e^{ik(x-ct)}\right) = 0$$

$$\hat{\phi}_1 \frac{\partial^2}{\partial x^2}\left(e^{ik(x-ct)}\right) + e^{ik(x-ct)} \frac{\partial^2 \hat{\phi}_1}{\partial y^2} = 0$$

$$
\begin{aligned}
-k^2\hat{\phi}_1 e^{ik(x-ct)} + e^{ik(x-ct)}\frac{\partial^2\hat{\phi}_1}{\partial y^2} &= 0 \\
e^{ik(x-ct)}\left[\frac{\partial^2\hat{\phi}_1}{\partial y^2} - k^2\hat{\phi}_1\right] &= 0 \\
\frac{d^2\hat{\phi}_1}{dy^2} &= k^2\hat{\phi}_1
\end{aligned}
$$

The above equation is a homogeneous second-order ODE with constant coefficients, and its general solution is

$$\hat{\phi}_1 = A\,e^{-ky} + C\,e^{ky}$$

and in a similar way, $\nabla^2\phi_2' = 0$ gives

$$\hat{\phi}_2 = D\,e^{-ky} + B\,e^{ky}$$

where A, B, C, and D are integration constants. It can be seen that to satisfy the boundary conditions (14), the constants C and D must be zero. Hence, the amplitude functions are of the forms:

$$
\begin{aligned}
\hat{\phi}_1 &= A\,e^{-ky} \\
\hat{\phi}_2 &= B\,e^{ky}
\end{aligned}
\tag{19}
$$

Substitution of equation (19) into the equation (18) for normal modes for ϕ_1' and ϕ_2' gives (equation for η is repeated below for convenience)

$$\eta = \hat{\eta}\,e^{ik(x-ct)} \tag{20}$$

$$\phi_1' = A\,e^{-ky}e^{ik(x-ct)} \qquad \phi_2' = B\,e^{ky}e^{ik(x-ct)} \tag{21}$$

Substituting normal modes (20) and (21) into the interface kinematic conditions (15) yields

$$
\begin{aligned}
\frac{\partial}{\partial y}\left(A\,e^{-ky}e^{ik(x-ct)}\right) &= \frac{\partial}{\partial t}\left(\hat{\eta}\,e^{ik(x-ct)}\right) + U_1\frac{\partial}{\partial x}\left(\hat{\eta}\,e^{ik(x-ct)}\right) \\
-kA\,e^{-ky}e^{ik(x-ct)} &= -ikc\hat{\eta}\,e^{ik(x-ct)} + ikU_1\hat{\eta}\,e^{ik(x-ct)} \\
A\,e^{-ky} &= -i\hat{\eta}\,(U_1-c)
\end{aligned}
$$

The above equation, when evaluated at $y=0$, provides an equation for A. A similar equation for B can also be obtained using the above procedure. Thus, we have

$$
\begin{aligned}
A &= -i\,(U_1-c)\,\hat{\eta} \\
B &= i\,(U_2-c)\,\hat{\eta}
\end{aligned}
\tag{22}
$$

The final equation is obtained by substituting equation (20) and (21) into dynamic interface condition (Bernoulli equation) (16). This gives

$$
\begin{aligned}
\rho_1\left(\frac{\partial\phi_1'}{\partial t} + U_1\frac{\partial\phi_1'}{\partial x} + g\eta\right) &= \rho_2\left(\frac{\partial\phi_2'}{\partial t} + U_2\frac{\partial\phi_2'}{\partial x} + g\eta\right) \\
\rho_1 e^{ik(x-ct)}\left(-ikcA\,e^{-ky} + ikU_1A\,e^{-ky} + g\hat{\eta}\right) &= \rho_2 e^{ik(x-ct)}\left(-ikcB\,e^{ky} + ikU_2B\,e^{ky} + g\hat{\eta}\right)
\end{aligned}
$$

At $y = 0$,

$$\rho_1 \left[ikA\left(U_1 - c\right) + g\hat{\eta}\right] = \rho_2 \left[ikB\left(U_2 - c\right) + g\hat{\eta}\right]$$

Substituting for A and B to obtain

$$\rho_1 \left[k\left(U_1 - c\right)^2 \hat{\eta} + g\hat{\eta}\right] = \rho_2 \left[-k\left(U_2 - c\right)^2 \hat{\eta} + g\hat{\eta}\right]$$

Rearranging

$$\rho_1 k\left(U_1 - c\right)^2 + \rho_2 k\left(U_2 - c\right)^2 = g\left(\rho_2 - \rho_1\right) \tag{23}$$

Equation (23) gives the eigenvalue relation for $c(k)$. Solving this equation for the complex wave speed:

$$c = c_r + ic_i = \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \left[\frac{g}{k}\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2} - \rho_1\rho_2\left(\frac{U_1 - U_2}{\rho_1 + \rho_2}\right)^2\right]^{1/2} \tag{24}$$

It can be seen that the wave speed c is real $(c_i = 0)$ if term within the square root is positive. Hence, both solutions are neutrally stable if

$$\frac{g}{k}\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2} \geq \rho_1\rho_2\left(\frac{U_1 - U_2}{\rho_1 + \rho_2}\right)^2$$

$$k \leq \frac{g}{\rho_1\rho_2}\frac{\rho_2^2 - \rho_1^2}{\left(U_1 - U_2\right)^2} \tag{25}$$

which gives the stable waves of the system. On the other hand, the flow (solution) is unstable if $c_i > 0$. That is,

$$k > \frac{g}{\rho_1\rho_2}\frac{\rho_2^2 - \rho_1^2}{\left(U_1 - U_2\right)^2} \tag{26}$$

Equation (24) shows that for each amplified (unstable) solution there exists an associated damped (stable) solution. This behavior is due to the fact that the coefficients of the perturbation differential equation and the boundary conditions are all real. If $U_1 \neq U_2$, then one can always find a large enough k that satisfies the requirement for instability. Because all wavelengths must be allowed in an instability analysis, we can say that the flow is always unstable to short waves if $U_1 \neq U_2$.

It is instructive to note that the dispersion relation of waves at the interface between two immiscible liquids of different densities can be obtained from equation (24) by setting $U_1 = U_2 = 0$:

$$c = \sqrt{\frac{g}{k}\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}} \tag{27}$$

or the circular frequency

$$\omega = \sqrt{gkAt} \tag{28}$$

where $At = (\rho_2 - \rho_1)/(\rho_1 + \rho_2)$, is the *Atwood number*.

Consider now the special case of a vortex sheet, i.e., the flow of a homogeneous fluid ($\rho_1 = \rho_2$) with a velocity discontinuity. Setting $\rho_1 = \rho_2 = \rho$ in equation (24) gives

$$c = c_r + ic_i = \rho^2 \frac{U_1 + U_2}{2\rho^2} \pm \left[-\rho^2 \frac{(U_1 - U_2)^2}{4\rho^2} \right]^{1/2}$$
$$c = c_r + ic_i = \frac{1}{2}(U_1 + U_2) \pm i\frac{1}{2}|U_1 - U_2| \qquad (29)$$

As we have discussed earlier, a flow of vortex sheet with $c_i > 0$ is unstable. Thus the vortex sheet (sheer layer) of a velocity jump in the interface is unstable for all wavelengths.

It is also seen that the perturbation waves move with a phase velocity equal to the average velocity of the basic flow $\frac{1}{2}(U_1 + U_2)$. This must be true from symmetry considerations. In a frame of reference moving with the average velocity, the basic flow is symmetric and the wave therefore should have no preference between the positive and negative directions. The Kelvin–Helmholtz instability is caused by the destabilizing effect of friction, which overcomes the stabilizing effect of density stratification. The Kelvin–Helmholtz instability is commonly found in Earth's atmosphere and ocean (see figure below).

Figure 2: A Kelvin–Helmholtz instability rendered visible by clouds over Mount Duval in Australia (image taken from Wikipedia)

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