Method of Lagrange Multipliers

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Lagrange multiplier method is a technique for finding a maximum or minimum of a function $F(x,y,z)$ subject to a *constraint* (also called *side condition*) of the form $G(x,y,z) = 0$.

Geometric basis of Lagrange multiplier method can be explained if the functions are of two variables. So we start by trying to find the extreme values of $F(x,y)$ subject to a constraint of the form $G(x,y) = 0$. In other words, we seek the extreme values of $F(x,y)$ when the point (x,y) is restricted to lie on the level curve $G(x,y) = 0$. Figure blow shows this curve together with several level curves of $F(x,y) = c$, where c is a constant. To maximize $F(x,y)$ subject to $G(x,y)$ is to find the largest value of c such that the level curve, $F(x,y) = c$, intersects $G(x,y) = 0$. It appears from figure that this happens when these curves just touch each other, that is, when they have a common tangent line. This means that the normal lines at the point (x_0,y_0) where they touch are identical. So the gradient vectors are parallel. That is

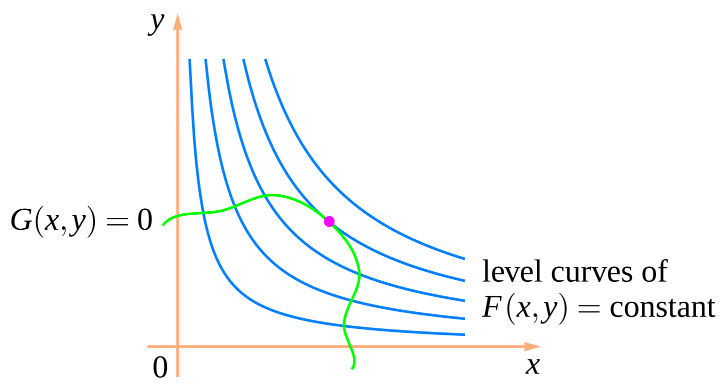

Figure 1: The four possible cases of varying end points in the direction of y.

$$\nabla F(x_0,y_0) = -\lambda \nabla G(x_0,y_0)$$

for some scalar λ. The scalar parameter λ is called a Lagrange multiplier. The procedure based on the above equation is as follows. We have from chain rule,

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0, \qquad dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy = 0$$

Multiplying the second equation by λ and add to first equation yields

$$\left(\frac{\partial F}{\partial x} + \lambda\frac{\partial G}{\partial x}\right)dx + \left(\frac{\partial F}{\partial y} + \lambda\frac{\partial G}{\partial y}\right)dy = 0$$

By choosing λ to satisfy

$$\frac{\partial F}{\partial x} + \lambda \frac{\partial G}{\partial x} = 0,$$

for example, so that

$$\frac{\partial F}{\partial y} + \lambda \frac{\partial G}{\partial y} = 0$$

As can be seen, the above two equations are components of the vector equation

$$\nabla F - \lambda \nabla G = 0 \tag{1}$$

Thus, the maximum and minimum values of $F(x,y)$ subject to the constraint $G(x,y) = 0$ can be found by solving the following set of equations

$$\begin{aligned} \frac{\partial F}{\partial x} + \lambda \frac{\partial G}{\partial x} &= 0 \\ \frac{\partial F}{\partial y} + \lambda \frac{\partial G}{\partial y} &= 0 \\ G(x,y) &= 0 \end{aligned} \tag{2}$$

This is a system of three equations in the three unknowns x, y, and λ, but it is not necessary to find explicit values for λ.

If the function to be extremized F and the side condition G are function of three independent variables x, y, and z, the following system of equation is solved to obtain the minimum or maximum value of F.

$$\begin{aligned} \frac{\partial F}{\partial x} + \lambda \frac{\partial G}{\partial x} &= 0 \\ \frac{\partial F}{\partial y} + \lambda \frac{\partial G}{\partial y} &= 0 \\ \frac{\partial F}{\partial z} + \lambda \frac{\partial G}{\partial z} &= 0 \\ G(x,y,z) &= 0 \end{aligned} \tag{3}$$

This is a system of four equations in the four unknowns x, y, z, and λ.

Example 1

A rectangular box without a lid is to be made from $27\,\mathrm{m}^2$ of cardboard. Find the maximum volume of such a box.

Method I – We first, solve this relatively simple problem without using Lagrange multiplier. Let the length, width, and height of the box (in meters) be x, y, and z. Then the volume of the box is

$$V = xyz$$

We can express V as a function of just two variables x and y by using the fact that the area of the five sides of the box is

$$xy + 2yz + 2xz = 27$$

Solving this equation for z, we get

$$z = \frac{27 - xy}{2(x+y)}$$

so that

$$V = xy\frac{27 - xy}{2(x+y)} = \frac{27xy - x^2y^2}{2(x+y)}$$

We compute the partial derivatives:

$$\frac{\partial V}{\partial x} = \frac{y^2(27 - 2xy - x^2)}{2(x+y)^2} \qquad \frac{\partial V}{\partial y} = \frac{x^2(27 - 2xy - y^2)}{2(x+y)^2}$$

If V is a maximum, then $\partial V/\partial x = \partial V/\partial x = 0$, but $x = 0$ or $y = 0$ gives $V = 0$, so we must solve the equations

$$27 - 2xy - x^2 = 0 \qquad 27 - 2xy - y^2 = 0$$

These equations imply that $x = y$, it may be noted the both x and y must be positive here. Putting $y = x$ in one of these equations, we get $27 - 3x^2 = 0$, which gives $x = 3$, $y = 3$, and $z = 1.5$. Thus the maximum volume occurs at $x = 3$, $y = 3$, and $z = 1.5$, so that the maximum volume of the box is $13.5\,\mathrm{m}^3$.

Method II – Here we wish to maximize

$$V = xyz$$

subject to the constraint

$$G(x,y,z) = xy + 2yz + 2xz - 27 = 0$$

Using the method of Lagrange multipliers, we look for values of x, y, z, and λ such that

$$\frac{\partial V}{\partial x} + \lambda\frac{\partial G}{\partial x} = 0 \qquad \frac{\partial V}{\partial y} + \lambda\frac{\partial G}{\partial y} = 0 \qquad \frac{\partial V}{\partial z} + \lambda\frac{\partial G}{\partial z} = 0$$
$$xy + 2yz + 2xz = 27$$

which become

$$yz + \lambda(y + 2z) = 0 \tag{4}$$
$$xz + \lambda(x + 2z) = 0 \tag{5}$$
$$xy + \lambda(2y + 2x) = 0 \tag{6}$$
$$xy + 2yz + 2xz = 27 \tag{7}$$

To solve this systems of equations in a convenient manner, we multiply the equation (4) by x, (5) by y, and (6) by z, then the left sides of these equations will be identical. Thus, we have

$$xyz = -\lambda(xy + 2xz) \tag{8}$$
$$xyz = -\lambda(xy + 2yz) \tag{9}$$
$$xyz = -\lambda(2yz + 2xz) \tag{10}$$

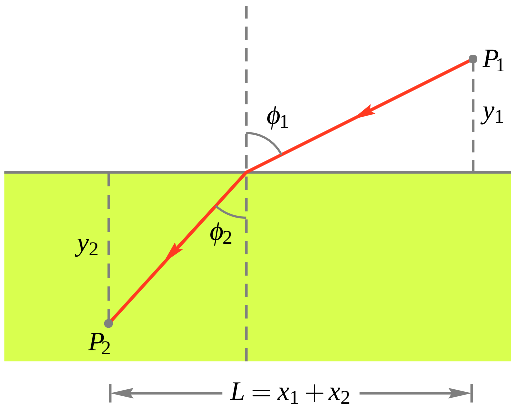

Figure 2: Illustration of Snell's law

We observe that $\lambda \neq 0$ because $\lambda = 0$ would imply $xy = yz = xz = 0$ and this would contradict the equation (7). Therefore, from equations (8) and (9), we have $xz = yz$. Since z cannot be zero, we have $x = y$. From equations (9) and (10), we have $y = 2z$. If we now put $x = y = 2z$ in equation (7), we get

$$12z^2 = 27$$

Since x, y, and z are all positive, we therefore have $z = 1.5$ and so $x = 3$ and $y = 3$.

Example 2

Here we will demonstrate how Lagrange multiplier method can be used for proving Snell's law. In the case of the inhomogeneous optical medium consisting of two homogeneous media in which the speed of light is piecewise constant. Suppose that the light travels from a point $P_1(x_1, y_1)$, with a constant speed v_1, in a homogenous medium M_1 to a point $P_2(x_2, y_2)$, with a constant speed v_2, in another homogeneous medium M_2. The two media are separated by the line $y = y_0$.

The time of transit of light is given by the geometry as

$$T = \frac{y_1/\cos\phi_1}{v_1} + \frac{y_2/\cos\phi_2}{v_2}$$

and is then subject to the geometrical constraint that

$$L = x_1 + x_2 = y_1 \tan\phi_1 + y_2 \tan\phi_2$$

Applying the condition (2)

$$\frac{\partial T}{\partial \phi_1} + \lambda \frac{\partial L}{\partial \phi_1} = \frac{y_1}{v_1} \sec\phi_1 \tan\phi_1 + \lambda y_1 \sec^2\phi_1 = 0$$
$$\frac{\partial T}{\partial \phi_2} + \lambda \frac{\partial L}{\partial \phi_2} = \frac{y_2}{v_2} \sec\phi_2 \tan\phi_2 + \lambda y_2 \sec^2\phi_2 = 0$$

These give as the only solution

$$\sin\phi_1 = -\lambda v_1 \qquad \sin\phi_2 = -\lambda v_2$$

or

$$\frac{\sin\phi_1}{v_1} = \frac{\sin\phi_2}{v_2}$$

where the angles are measured with respect to the normal of the boundary between the two media.