# An Exact Solution of Navier–Stokes Equation

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The principal difficulty in solving the Navier–Stokes equations (a set of nonlinear partial differential equations) arises from the presence of the nonlinear convective term $(\overline{V} \cdot \nabla)\overline{V}$. Since there are no general analytical methods for solving nonlinear partial differential equations exist, each problem must be considered individually. For most practical flow problems, convective acceleration of fluid particles cannot be ignored. However, in general, exact solutions are possible only when the nonlinear terms vanishes identically. There are a few special cases for which the convective acceleration vanish because of the nature of the geometry of the flow system. In these cases exact solutions are usually possible and below we consider one of such problems.

## Pipe Flow Induced by Movement of Wall

Here we consider an infinitely long horizontal circular pipe filled with a Newtonian fluid of density $\rho$ and viscosity $\mu$. For $t < 0$ both pipe and fluid are at rest. At time $t = 0$, the pipe impulsively starts moving in the axial direction with a uniform velocity $U$. As a consequence of this, the fluid movement is induced in the axial direction as sketched in figure 1. The momentum equation for incompressible flow in cylindrical coordinate system

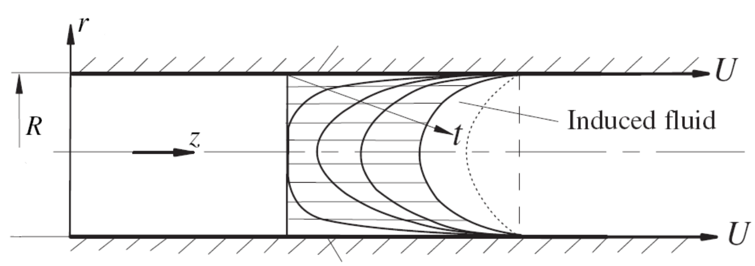


Figure 1: Fluid flow in a pipe induced by the motion of pipe walls.

is given by

$$\rho\left(\frac{Du_r}{Dt}-\frac{u_\theta^2}{r}\right)=\rho g_r-\frac{\partial p}{\partial r}+\mu\left(\nabla^2 u_r-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{u_r}{r^2}\right)$$
$$\rho\left(\frac{Du_\theta}{Dt}+\frac{u_r u_\theta}{r}\right)=\rho g_\theta-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left(\nabla^2 u_\theta+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r^2}\right)$$
$$\rho\frac{Du_z}{Dt}=\rho g_z-\frac{\partial p}{\partial z}+\mu\nabla^2 u_z$$

For the conditions of the present unsteady parallel flows in the absence of body forces, many terms disappear and the axial velocity $u_z$ is the only nonzero velocity component. Therefore the first two equations become trivial and the last equation (axial momentum equation) reduces to

$$\rho\frac{\partial u_z}{\partial t}=\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial\theta^2}\right] \tag{1}$$

Since the flow is rotationally symmetric, equation (1) reduces to

$$\frac{\partial u_z}{\partial t}=\nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) \tag{2}$$

The initial and boundary conditions are

$$\begin{aligned} u_z(r,0)&=0 \qquad \text{(initial condition)}\\ u_z(0,t)&\quad \text{is bounded}\\ u_z(R,t)&=U \qquad \text{(no-slip condition)}\end{aligned} \tag{3}$$

For convenience, we introduce the following dimensionless quantities:

$$u=\frac{U-u_z}{U}$$
$$\eta=\frac{r}{R}$$
$$\tau=\frac{\nu t}{R^2}$$

Thus the differential equation (2) can be written in dimensionless quantities as follows:

$$-\nu\frac{U}{R^2}\frac{\partial u}{\partial\tau}=-\nu\frac{U}{R^2}\left[\frac{\partial^2 u}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial u}{\partial\eta}\right]$$
$$\frac{\partial u}{\partial\tau}=\frac{\partial^2 u}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial u}{\partial\eta} \tag{4}$$

Corresponding dimensionless initial and boundary conditions are

$$\begin{aligned} u(\eta,0)&=1 \qquad \text{(initial condition)}\\ u(0,\tau)&\quad \text{is bounded}\\ u(1,\tau)&=0 \qquad \text{(no-slip condition)}\end{aligned} \tag{5}$$

Note that the new boundary conditions are homogeneous.

Although the problem defined by (4) and (5) is time dependent, it is linear in $u$ and confined to the bounded spatial domain, $0 \leq \eta \leq 1$. Thus it can be solved by the method of separation of variables. In this method we first find a set of *eigensolutions* that satisfy the differential equation (5) and the boundary condition at $\eta = 0$ and $\eta = 1$; then we determine the particular sum of those eigensolutions that also satisfies the initial condition at $\tau = 0$. The problem (4) and (5) comprises one example of the general class of so-called *Sturm–Liouville problems* for which an extensive theory is available that ensures the existence and uniqueness of solutions constructed by means of eigenfunction expansions by the method of separation of variables. We begin with the basic hypothesis that a solution of (5) exists in the separable form and choose the following ansatz:

$$u(\eta, \tau) = F(\eta)\, G(\tau) \tag{6}$$

Substituting this ansatz into equation (4) to obtain

$$F\frac{dG}{d\tau} = G\frac{d^2F}{d\eta^2} + \frac{G}{\eta}\frac{dF}{d\eta} \tag{7}$$

As $G$ depends only on $\tau$ and $F$ only on $\eta$, by separation of variables the following ordinary differential equations for $G$ and $F$ result:

$$\frac{1}{G}\frac{dG}{d\tau} = -\lambda^2 \tag{8}$$

$$\frac{1}{F}\frac{d^2F}{d\eta^2} + \frac{1}{F}\frac{1}{\eta}\frac{dF}{d\eta} = -\lambda^2 \tag{9}$$

Hence the original problem governed by a PDE has now been transformed into two related auxiliary problems governed by ODEs. The choice of a negative constant $\lambda^2$ is due to the fact that the solution will decay to zero as time increases, i.e., $u \to 0$ as $\tau \to \infty$. The solution for the differential equation (8) can be derived by integration:

$$G = c_0\, e^{-\lambda^2 \tau} \tag{10}$$

where $c_0$ is an integration constant to be determined. In order to determine the solution of the differential equation for $F(\eta)$, equation (9) can be written as follows:

$$\frac{d^2F}{d\eta^2} + \frac{1}{\eta}\frac{dF}{d\eta} + \lambda^2 F = 0$$

or

$$\eta^2\frac{d^2F}{d\eta^2} + \eta\frac{dF}{d\eta} + \lambda^2\eta^2 F = 0 \tag{11}$$

or, on introducing a change of independent variables,

$$y = \lambda \eta$$

we obtain

$$y^2 \frac{d^2F}{dy^2} + y\frac{dF}{dy} + y^2F = 0 \tag{12}$$

This is Bessel differential equation of order 0. It has two linearly independent solutions,

$$J_0(y) \qquad \text{and} \qquad Y_0(y)$$

which are known as Bessel functions of the first and second kinds of order 0. Thus the most general solution of equation (12) can be written as

$$F(y) = c_1 J_0(y) + c_2 Y_0(y) \tag{13}$$

A plot showing the behavior of these two functions is shown in figure (2). Both oscillate

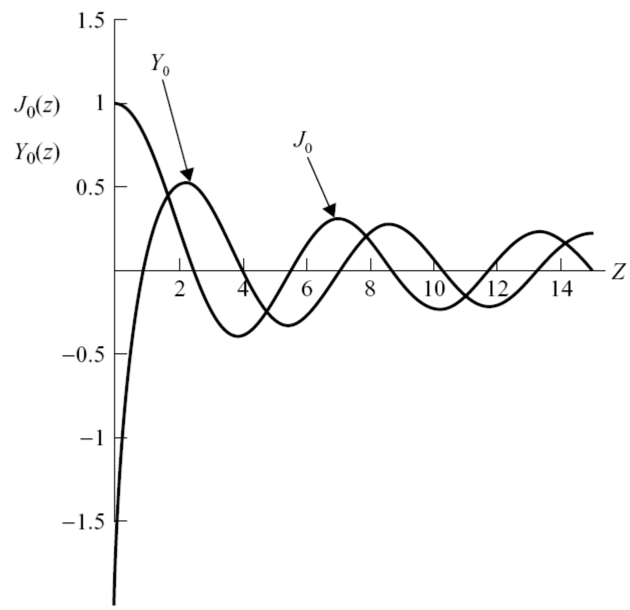


Figure 2: The Bessel functions of the first and second kinds of order 0.

back and forth across zero, but $Y_0(y) \to \infty$ as $y \to 0$.

Substituting equations (10) and (13) into (6), we obtain the general solution of the form

$$u(y, \tau) = e^{-\lambda^2 \tau}[c_1 J_0(y) + c_2 Y_0(y)] \tag{14}$$

Note that the constant $c_0$ has been dropped since it is redundant here. The solution is bounded at $y = 0$. This condition is satisfied only when the constant $c_2 = 0$. Hence, the general solution (14) (that is bounded at $y = 0$) takes a form:

$$u(y, \tau) = c_1 J_0(y) e^{-\lambda^2 \tau} \tag{15}$$

Therefore, the solution $u$, after substituting $y = \lambda\eta$, may be written as

$$u_n(\eta,\tau) = A_n J_0(\lambda_n \eta)\, e^{-\lambda_n^2 \tau} \tag{16}$$

The subscript $n$ is added in anticipation of the fact that there is an infinite, but discrete, set of values possible for $\lambda$ such that the general solution (15), satisfies the boundary condition $u = 0$ at $\eta = 1$. This set of values of $\lambda = \lambda_n$ is known as the eigenvalues of the problem, and the corresponding $u_n$ are called the eigenfunctions.

To determine the eigenvalues $\lambda_n$, we apply the boundary condition at $\eta = 1$ to equation (16), that is,

$$0 = A_n J_0(\lambda_n)\, e^{-\lambda_n^2 \tau} \qquad \text{for all} \quad \tau$$

Since setting $A_n = 0$ would results in a trivial solution, one must require

$$J_0(\lambda_n) = 0 \tag{17}$$

for the non-trivial solution. Therefore, one obtains multiple values of $\lambda_n$ (eigenvalues) that satisfy the boundary conditions at the wall. Clearly these eigenvalues are equal to the infinite set of zeroes of the Bessel function of order zero, $J_0(z)$. Referring to figure 2, we have denoted those zeroes as $s_n$, with the first crossing for the smallest value of $z$ being $s_1$, namely,

$$\lambda_n = s_n, \qquad n = 1, 2, 3, \ldots, \infty \tag{18}$$

and their values are obtained as

$$s_n = 2.405, 5.520, 8.654, 11.792, 14.931, 18.071, 21.212, 24.353, 27.494 \tag{19}$$

Each of the solutions of $\lambda_n$ now constitutes an individual solution. Considering the linearity of the governing equation and boundary conditions (4) and (5), the complete solution for $u_n(\eta,\tau)$ is obtained by linear superposition:

$$u = \sum_{n=1}^{\infty} u_n(\eta,\tau) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 \tau} J_0(\lambda_n \eta) \tag{20}$$

where $A_n$ are arbitrary, constant coefficients. Equation (20) is called the *Fourier–Bessel series*. This solution satisfies the differential equation (4) and the boundary condition $u = 0$ at $\eta = 1$ for any choice of the constant coefficients $A_n$.

The final step is to choose the $A_n$ so that $u(\eta,\tau)$ satisfies the initial condition $u(\eta, 0) = 1$. The general Sturm–Liouville theory guarantees that the eigenfunctions (16) form a complete set of orthogonal functions. Thus it is possible to express the smooth initial condition 1 by means of the Fourier–Bessel series (20) with $\tau = 0$, that is,

$$u(\eta, 0) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n \eta) = A_1 J_0(\lambda_1 \eta) + A_2 J_0(\lambda_2 \eta) + \ldots A_n J_0(\lambda_n \eta) + \ldots = 1 \tag{21}$$

To determine the $A_n$, we will take advantage of orthogonality properties of $J_0$:

$$\int_0^1 J_0(\lambda_m\eta)\,J_0(\lambda_n\eta)\,\eta\,d\eta = \begin{cases} \frac{1}{2}J_1^2(\lambda_n) & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \tag{22}$$

where $J_1$ is the Bessel function of first kind of order 1. Multiplying both sides of (21) by $\eta J_0(\lambda_m\eta)$ and integrating over $\eta$ from 0 to 1, we obtain

$$\sum_{n=1}^{\infty} A_n \int_0^1 \eta\,J_0(\lambda_m\eta)\,J_0(\lambda_n\eta)\,d\eta = \int_0^1 \eta\,J_0(\lambda_m\eta)\,d\eta$$

Due to the orthogonality property (22), the only nonzero term on the left hand side is that for $m = n$ hence,

$$A_n = \frac{\int_0^1 \eta\,J_0(\lambda_n\eta)\,d\eta}{\int_0^1 \eta\,J_0^2(\lambda_n\eta)\,d\eta} = \frac{\int_0^1 \eta\,J_0(\lambda_n\eta)\,d\eta}{\frac{1}{2}J_1^2(\lambda_n)} \tag{23}$$

For evaluating the numerator of (23) we make use of the following property of Bessel function:

$$\int \eta^{p+1} J_p(\lambda\eta)\,d\eta = \frac{1}{\lambda}\eta^{p+1}J_{p+1}(\lambda\eta) \tag{24}$$

Therefore

$$\int \eta\,J_0(\lambda_n\eta)\,d\eta = \frac{1}{\lambda_n}\eta J_1(\lambda_n\eta) \tag{25}$$

and thus, the numerator of equation (23) becomes

$$\int_0^1 \eta\,J_0(\lambda_n\eta)\,d\eta = \frac{1}{\lambda_n}J_1(\lambda_n) \tag{26}$$

Substitution of equation (26) into (23) yields an expression for $A_n$:

$$A_n = \frac{\frac{1}{\lambda_n}J_1(\lambda_n)}{\frac{1}{2}J_1^2(\lambda_n)} = \frac{2}{\lambda_n}\left[J_1(\lambda_n)\right]^{-1} \tag{27}$$

Thus for the velocity distribution according to (20), the following expression results:

$$u(\eta,\tau) = 2\sum_{n=1}^{\infty} \frac{1}{\lambda_n J_1(\lambda_n)}\,e^{-\lambda_n^2\tau}\,J_0(\lambda_n\eta) \tag{28}$$

The above Fourier–Bessel series has the property of converging very quickly when the dimensionless time $\tau = \nu t/R^2$ is large. On the other hand, the convergence is slow when $\tau$ is small. Reverting to dimensional variables, we can express the solution of the full, original problem in terms of the axial velocity profile:

$$u_z(r,t) = U\left[1 - 2\sum_{n=1}^{\infty} \frac{1}{\lambda_n J_1(\lambda_n)}\,e^{-\lambda_n^2\nu t/R^2}\,J_0\left(\lambda_n\frac{r}{R}\right)\right] \tag{29}$$

Obviously, as $t \to \infty$, this solution reverts to the steady-state uniform flow profile. To obtain other details of this velocity profile, it is necessary to evaluate the infinite series numerically for each value of $t$ and $r$. A typical numerical example of the results is shown in figure 3, where $u_z$ has been plotted versus $r$ for several values of $t$.

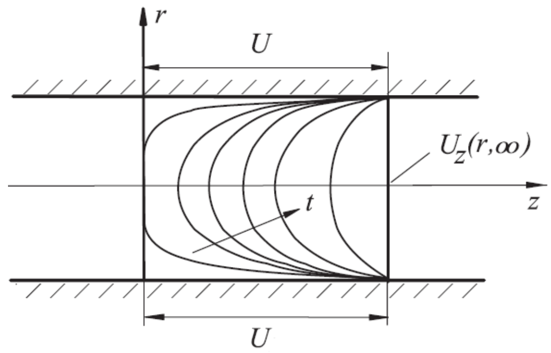


Figure 3: Transient velocity profiles in a pipe induced by the motion of the pipe walls.

## Shear stress distribution

The shear stress distribution in the flowfield can be easily obtained from the velocity distribution as follows

$$\tau_{rx} = \mu\left(\frac{\partial u_r}{\partial x} + \frac{\partial u_z}{\partial r}\right) = \mu\frac{du_z}{dr}$$

In dimensionless form

$$\bar{\tau}_{rx} = \frac{du}{d\eta}$$

where dimensionless shear stress $\bar{\tau}_{rx} = \frac{\tau_{rx}}{\mu U/R}$. For evaluating the derivative of velocity profile, we make use of the following property of Bessel function:

$$\frac{d}{d\eta}\left[\eta^{-p}J_p(\lambda\eta)\right] = -\lambda\eta^{-p}J_{p+1}(\lambda\eta) \tag{30}$$

Therefore

$$\frac{d}{d\eta}\left[J_0(\lambda\eta)\right] = -\lambda J_1(\lambda\eta) \tag{31}$$

and thus the shear stress profile is given by

$$\bar{\tau}_{rx} = \frac{du}{d\eta} = -2\sum_{n=1}^{\infty}\left[J_1(\lambda_n)\right]^{-1} e^{-\lambda_n^2\tau} J_1(\lambda_n\eta) \tag{32}$$

For the value of wall shear stress at the pipe wall, i.e. at $\eta = 1$, one obtains

$$\bar{\tau}_{\text{wall}} = \bar{\tau}_{rx}\big|_{\eta=1} = \left.\frac{du}{d\eta}\right|_{\eta=1} = -2\sum_{n=1}^{\infty} e^{-\lambda_n^2\tau} \tag{33}$$

Reverting to dimensional variables, we can express the wall shear stress in terms of original variables:

$$\tau_{\text{wall}} = \left.\frac{du_z}{dr}\right|_{r=R} = -\frac{2\mu U}{R}\sum_{n=1}^{\infty} e^{-\lambda_n^2\nu t/R^2} \tag{34}$$

It can be seen that the wall shear stress has a finite value, even for time $t = 0$. This is a surprising result when comparing to other cases where flow is induced by impulsively motion of the boundary. Note also that the wall shear stress $\tau_{\text{wall}} \to 0$ as $t \to \infty$. This is obvious because the steady-state velocity profile is uniform.