Lecture Notes on Partial Differential Equations

Dr. E. Natarajan

IIST Lecture Notes Series-2

Government of India
Department of Space
Indian Institute of Space Science and Technology
Valiamala P.O, Thiruvananthapuram-695547
December, 2012
Lecture Notes on Partial Differential Equations
Dr. E. Natarajan

Assistant Professor, Department of Mathematics
Indian Institute of Space Science and Technology,
Valiamala P.O, Thiruvananthapuram, India.

[For internal circulation only.]

Published by:
Government of India
Department of Space
Indian Institute of Space Science and Technology
Deemed to be University under Section 3 of the UGC Act, 1956
Valiamala P.O, Thiruvananthapuram-695547
Acknowledgement

This lecture notes is based on my teaching B. Tech students of IIST for the course MA 221 for several semesters. The main purpose of this notes is to give students material which is mostly self explanatory with deep conceptual backend suited for undergraduates.

I thank Dr.K.S.Dasgupta Director, IIST who motivated me to prepare this material. I thank Library and Information services, IIST for support. I would thank all the members of Department of Mathematics specially Shri. Abdul Karim.

Hope this will go into the minds of young students and not into the unread shelf of library.
Contents

1 Modeling partial differential equations 1

1.1 Introduction ........................................... 1
1.2 Vibrating string problem ............................... 2
1.3 Heat equation .......................................... 5
1.4 Well-posed PDE ....................................... 6
1.5 Solution of PDE’s by Fourier transform .......... 7

2 Second order linear PDE’s 11

2.1 Classification .......................................... 11
2.1.1 Definitions ......................................... 11
2.2 Canonical forms ....................................... 13
2.2.1 Canonical form of hyperbolic PDE ............ 14
2.2.2 Canonical form of parabolic PDE ......... 16
Chapter 1

Modeling partial differential equations

1.1 Introduction

Partial differential equations is a relation between an unknown function and its partial derivatives

\[ F(x_1, x_2, \ldots, x_n, u_{x_1}, u_{x_2}, \ldots, u_{x_11}) = 0 \]  

where \( x_1, x_2, \ldots, x_n \) are independent variables and \( u(x_1, x_2, \ldots, x_n) \) dependent variable \( u_{x_i} = \frac{\partial u}{\partial x_i} \)

- Order of the partial differential equation is the order of the highest partial derivative. \( u_{tt} - u_{xx} = f(x, t) \) is a second order PDE whereas \( u_t + u_{xxxx} = 0 \) is a fourth order PDE.
Figure 1.1: Vertical displacement of string

- An equation is called linear if in Eq(1.1.1) \( F(.) \) is a linear function of the unknown function \( u \) and its partial derivatives. For example \( x^2 u_x + x y u_y + \sin(x^2 + y^2) u = x^3 \) is a linear equation whereas \( u_{xx}^2 + u_{yy}^2 = 0 \) is a non-linear equation. An equation is quasilinear if it is linear in the highest partial derivative \( u_{xx} + u_{yy} = |\nabla u|^2 u \) and semilinear if it is non-linear in its unknown function \( u_{xx} + u_{yy} = u^3 \).

1.2 Vibrating string problem

Let \( \psi(x, t) \) measures the vertical displacement of the string from equilibrium. Length of this piece \( \sqrt{\Delta x^2 + \Delta \psi^2} \) where \( \Delta \psi = \psi(x + \Delta x, t) - \psi(x, t) \) with Density \( \rho \) (mass per unit length) is the mass of the string particle.

\[
\rho \sqrt{\Delta x^2 + \Delta \psi^2} = \rho \Delta x \sqrt{1 + O\left(\frac{\Delta \psi}{\Delta x}\right)^2} \quad (1.2.1)
\]
• Displacement of the string is slight, so terms of order 2 for $\psi$ and $\frac{\partial \psi}{\partial x}$ are neglected.

Mass $\approx \rho \Delta x$ and acceleration in the vertical direction is $\frac{\partial^2 \psi}{\partial t^2}$. Force acting on the string particle is Tension $T$. This acts parallel to the string i.e., tangentially. Tension is assumed to be uniform. This gives

$$T(x, t) = T \frac{\dot{i} + \psi'(x) \dot{j}}{\sqrt{1 + (\psi'(x))^2}}$$

(1.2.2)

Vertical component, $\uparrow T(x, t) = \frac{T \frac{\partial \psi}{\partial x}}{1 + (\psi'(x))^2} \approx T \frac{\partial \psi}{\partial x}$

Net vertical tensile force is, $\uparrow T(x + \Delta x, t) - \uparrow T(x, t) \approx \frac{T \partial^2 \psi}{\partial x^2} + \mathcal{O}(\Delta x^2)$

Using Newton’s second law

$$T \frac{\partial^2 \psi}{\partial x^2} \Delta x + \mathcal{O}(\Delta x^2) = \rho \Delta x \frac{\partial^2 \psi}{\partial t^2}$$

(1.2.3)

Dividing by $\rho \Delta x$ and letting $\Delta x \to 0$ we get $\frac{\partial^2 \psi}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 \psi}{\partial x^2} = c^2 \frac{\partial^2 \psi}{\partial x^2}$

where $c = \sqrt{\frac{T}{\rho}}$ (dimensions of velocity).

Let $f$ be twice-differentiable then $\psi(x, t) = f(x - ct)$ satisfies the wave equation. It is the wave propagating to the left at speed $c$ and $f(x + ct)$ is the wave propagating to the right at speed $c$ so the solution will be $\psi(x, t) = f(x - ct) + f(x + ct)$ where $f$ and $g$ are arbitrary.

In order to determine the motion completely $\psi(x, 0) = F(x)$ and
\[ \frac{\partial \psi}{\partial t}(x, 0) = G(x) \]
are given as initial displacement and initial velocity.

\[ \psi(x, t) = \frac{F(x - ct) + F(x + ct)}{2} + \frac{1}{2c} \int_{x - ct}^{x + ct} G(\xi) \, d\xi \quad (1.2.4) \]

This is known as the D’Alembert’s form of the solution. This says that to determine \( \psi(x, t) \) we need to know \( F \) and \( G \) within \( \pm ct \) distance.

Types of boundary condition for the string tied at left boundary:

- Dirichlet boundary condition \( \psi(0, t) = c \)
- Neumann boundary condition \( \frac{\partial \psi}{\partial x}(0, t) = d \)
- Robin boundary condition \( \alpha \psi(0, t) + \beta \frac{\partial \psi}{\partial x}(0, t) = l \)
1.3 Heat equation

Consider heat flowing through a uniform thin rod. Experiment shows that \( \Delta Q = c \rho u(x, t) \Delta V \) for a uniform cross section \( \Delta V = A \Delta x \) between small portion \( x \in [a, b] \) where \( c \) is the concentration, \( \rho \) is the density, \( u(x, t) \) is the temperature and \( \Delta V \) is the small volume element.

\[
Q(x, t) = \int_{a}^{b} c \rho u(x, t) A \, dx
\]

\[
\frac{dQ}{dt} = \int_{a}^{b} c \rho \frac{\partial u(x, t)}{\partial t} A \, dx
\]

This gives the rate at which heat accumulates in this segment. According to Newton’s Law of Cooling the rate of change of the temperature of an object is proportional to the temperature gradient. Also heat energy flows from warmer region to cooler region. This gives

\[
\frac{dQ}{dt} = \tilde{k} \left[ \text{Rate in} - \text{Rate out} \right] A
\]

\[
\frac{dQ}{dt} = \tilde{k} \left[ \frac{\partial u(b, t)}{\partial x} - \frac{\partial u(a, t)}{\partial x} \right] A
\]

In the integral form this gives

\[
\int_{a}^{b} \left( c \rho \frac{\partial u(x, t)}{\partial t} - \tilde{k} \frac{\partial^2 u}{\partial x^2} \right) A \, dx = 0
\]
Under suitable conditions it reduces to \( \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \) where \( k = \frac{\tilde{k}}{\rho c} \) is the thermal diffusivity.

### 1.4 Well-posed PDE

Any physical phenomena can be modeled and it gives rise to partial differential equation. Suppose motion of a pendulum is modeled as a differential equation, definitely basic problem has a sure motion whereas the modeled equation need not have solution. This doesn’t contradict with the physics of the problem but it says that the modeled equations are not sufficient enough to talk about the physics of the problem.

Jacques Hadamard came up with a set of conditions PDE has to satisfy so that the equation can undoubtedly talk about the physics of the problem and the PDEs satisfying those conditions are known as Well-posed problems. They are

(i) Existence of a solution

(ii) Uniqueness of the solution (there exists only one solution)

(iii) Solution is stable (continuous dependence of data)

We will illustrate the third point with an example.

\[
\begin{align*}
  u_t &= k u_{xx}, \quad -\infty < x < \infty \quad (1.4.1) \\
  u(x, 1) &= c \sin \left( \frac{x}{c\sqrt{k}} \right) \quad (1.4.2)
\end{align*}
\]
where $c$ is small. This can be solved by taking $u(x,t) = c \sin \left( \frac{x}{c \sqrt{k}} \right) g(t)$ with $g(1) = 1$. (Try it!). The solution is $u(x,t) = ce^{\left(1 - \frac{t}{c^2} \right)} \sin \left( \frac{x}{c \sqrt{k}} \right)$.

Note that when $c \to 0$ when $0 < t < 1$ the solution $u(x,t) \to \infty$. A small change in the initial condition leads to a large change in the solution thus killing the stability of the problem.

1.5 Solution of PDE’s by Fourier transform

Heat conduction in an infinite rod

$$u_t = \alpha^2 u_{xx}, \quad -\infty < x < \infty, \quad 0 < t < \infty$$
$$u(x,0) = f(x), \quad -\infty < x < \infty$$

Note: This problem can be solved by separation of variables if $f(x)$ is defined in finite interval or even if $f$ is defined in infinite interval provided if it is periodic. Suppose $f$ is defined in infinite interval and aperiodic like $f(x) = e^{-|x|}$. Let us take fourier transform of heat equation with respect to $x$.

$$F\{u_t\} = \alpha^2 F\{u_{xx}\}$$

$$\int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-i \omega x} \, dx = \alpha^2 (i \omega)^2 \hat{u}$$

$$\frac{d}{dt} \int_{-\infty}^{\infty} u(x,t) e^{-i \omega x} \, dx = -\alpha^2 \omega^2 \hat{u}$$

Under the assumption $(u \to 0, \, u_x \to 0, \, x \to \pm \infty)$

$$\frac{d\hat{u}}{dt} + \alpha^2 \omega^2 \hat{u} = 0$$
The solution is \( \hat{u}(x, t) = A(\omega) e^{-\alpha^2 \omega^2 t} \). We also need to transform the initial condition. \( F(u(x, 0)) = F(f(x)) \Rightarrow \hat{u}(\omega, 0) = \hat{f}(\omega) \). Using this in the solution we get \( A = \hat{f}(\omega) \). Finally we get

\[ u(x, t) = \frac{1}{2 \alpha \sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi) e^{-\frac{(x-\xi)^2}{4 \alpha^2 t}} d\xi \]

Case 1: If \( f \) is constant \( F \) then \( u(x, t) = F \) which violates the assumption \( u \to \infty, x \to \pm \infty \).

Case 2: \( f(x) = \begin{cases} F, & x > 0 \\ 0, & x < 0 \end{cases} = FH(x) \)

\[ u(x, t) = \frac{F}{2} \left[ 1 + \text{erf} \left( \frac{x}{2 \alpha \sqrt{t}} \right) \right] \]

where \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} dt \).

In a more formidable way our solution can be written as

\[ u(x, t) = \int_{-\infty}^{\infty} f(\xi) K(\xi - x; t) d\xi \]

where

\[ K(\xi - x; t) = \frac{e^{-\frac{(x-\xi)^2}{4 \alpha^2 t}}}{2 \alpha \sqrt{\pi t}} \]

is called as Kernel of the integral. If \( K(\xi - x; t) \to \delta(\xi - x) \) as \( t \to 0 \) then

\[ \lim_{t \to 0} u(x, t) = \lim_{t \to 0} \int_{-\infty}^{\infty} f(\xi) K(\xi - x; t) d\xi = f(x) \]

Let \( f(x) = \delta(x - x_0) \). Then

\[ u(x, t) = \int_{-\infty}^{\infty} \delta(\xi - x_0) K(\xi - x; t) d\xi = K(x_0 - x; t) = K(x - x_0; t) \]
If initial temperature distribution is $\delta(x - x_0)$ then $K(x - x_0; t)$ is the solution to the heat equation. Diffusion process smooths out the solution.

**Example 1.5.1.** $u_t = u_{xx}$, $-\infty < x < \infty$, $t > 0$, $u(x, 0) = F(x)$, $-\infty < x < \infty$. By solving this problem using Fourier transform we get

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} F(y) e^{-\frac{(x-y)^2}{4t}} dy$$

where $e^{\frac{(x-y)^2}{4t}} > 0$ $\forall x, y$. For $t$ however small $u$ depends on $F(x)$, $-\infty < x < \infty$ which means the speed of propagation is infinite. But energy cannot propagate faster than speed of light. This model of heat equation has a flaw.
Chapter 2

Second order linear PDE’s

2.1 Classification

Let us consider the general form of second order linear PDE’s in two variables

\[ A(x, y) u_{xx} + 2B(x, y) u_{xy} + C(x, y) u_{yy} = \Phi(x, y, u_x, u_y, u) \]  (2.1.1)

where the left hand side of Eq (2.1.1) denotes the principal part of the PDE.

2.1.1 Definitions

(a) Equation 2.1.1 is said to be hyperbolic if \( B^2 - AC > 0 \)

(b) Equation 2.1.1 is said to be parabolic if \( B^2 - AC = 0 \)

(c) Equation 2.1.1 is said to be elliptic if \( B^2 - AC < 0 \)
What do these classification do? Why is it needed?

These classifications can be related with equation of general conic section

\[ a x^2 + 2 b x y + c y^2 + d x + e y + f = 0 \]  

(2.1.2)

(a) Equation 2.1.2 represents eqn of hyperbola if \( b^2 - ac > 0 \)

(b) Equation 2.1.2 represents eqn of parabola if \( b^2 - ac = 0 \)

(c) Equation 2.1.2 represents eqn of ellipse if \( b^2 - ac < 0 \)

The need for this classification is any second order PDE’s in two variables can be reduced to either hyperbolic, parabolic or elliptic. Once the behaviour of the solutions of hyperbolic, parabolic and elliptic PDEs are known then classification to one of the three forms helps in understanding the problem apriori.

Examples:

\[ u_{tt} = c^2 u_{xx} \]  
where \( A = 1, B = 0, C = -c^2 \) gives \( B^2 - AC > 0 \)  
then the equation is said to be hyperbolic.

\[ u_t = k u_{xx} \]  
where \( A = 0, B = 0, C = -k \) gives \( B^2 - AC = 0 \)  
then the equation is said to be parabolic.

\[ u_{xx} + u_{yy} = 0 \]  
where \( A = 1, B = 0, C = 1 \) gives \( B^2 - AC < 0 \)  
then the equation is said to be elliptic.
2.2 Canonical forms

Any second order PDE in two variables can be reduced to canonical form of hyperbolic, parabolic and elliptic PDE’s under suitable transformation. Let us convert the PDE Equation 2.1.1 using the coordinate transformation $\xi(x, y), \eta(x, y)$. Also take $w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$

\[
\begin{align*}
    u_x &= w_\xi \xi_x + w_\eta \eta_x \\
    u_y &= w_\xi \xi_y + w_\eta \eta_y
\end{align*}
\]

Similarly compute $u_{xx}, u_{yy}$ and $u_{xy}$ in terms of $w_{\xi\xi}, w_{\xi\eta}$ and $w_{\eta\eta}$. After substitution of these terms into the Equation 2.1.1 gives

\[
a(\xi, \eta) w_{\xi\xi} + 2 b(\xi, \eta) w_{\xi\eta} + c(\xi, \eta) w_{\eta\eta} = \Psi(w_\xi, w_\eta, w, \xi, \eta)
\]

where

\[
\begin{align*}
    a(\xi, \eta) &= A \xi_x^2 + 2 B \xi_x \xi_y + C \xi_y^2 \\
    b(\xi, \eta) &= A \xi_x \eta_x + B (\xi_x \eta_y + \xi_y \eta_x) + C \xi_y \eta_y \\
    c(\xi, \eta) &= A \eta_x^2 + 2 B \eta_x \eta_y + C \eta_y^2
\end{align*}
\]

We need a justification that the form of the PDE remains invariant even after the coordinate transformation i.e., hyperbolic remains as hyperbolic, parabolic remains parabolic and vice versa. It can be observed that

\[
\begin{pmatrix}
    a & b \\
    b & c
\end{pmatrix} =
\begin{pmatrix}
    \xi_x & \xi_y \\
    \eta_x & \eta_y
\end{pmatrix} \begin{pmatrix}
    A & B \\
    B & C
\end{pmatrix} \begin{pmatrix}
    \xi_x & \eta_x \\
    \xi_y & \eta_y
\end{pmatrix}
\]

Taking the determinant on both sides gives

\[
b^2 - ac = (\xi_x \eta_y - \xi_y \eta_x)^2 (B^2 - AC)
\]
where \((\xi_x \eta_y - \xi_y \eta_x)^2\) is nothing but the square of the Jacobian \(J^2\).

The canonical form of the PDE after transformation reduces to

(i) \(w_{\xi \eta} = G(w_\xi, w_\eta, \xi, \eta)\) if it is hyperbolic.

(ii) \(w_{\xi \xi} = G(w_\xi, w_\eta, \xi, \eta)\) or \(w_{\eta \eta} = G(w_\xi, w_\eta, \xi, \eta)\) if it is parabolic.

(iii) \(w_{\xi \xi} + w_{\eta \eta} = G(w_\xi, w_\eta, \xi, \eta)\) if it is elliptic.

### 2.2.1 Canonical form of hyperbolic PDE

If \(A = C = 0\) then \(u_{xy} = \frac{\Phi(u_x, u_y, u, x, y)}{2B}\) in which case there is no need of transformation.

Let us assume \(A \neq 0\) and in order to make \(b^2 - ac > 0\) we take \(a = c = 0\). This gives

\[
\begin{align*}
A \xi_x^2 + 2B \xi_x \xi_y + C \xi_y^2 &= 0 \\
A \eta_x^2 + 2B \eta_x \eta_y + C \eta_y^2 &= 0
\end{align*}
\]

After factorizing

\[
\frac{1}{A} \left( A \xi_x + \left( B - \sqrt{B^2 - AC} \xi_y \right) \right) \\
\left( A \xi_x + \left( B - \sqrt{B^2 + AC} \xi_y \right) \right) = 0
\]

14
\begin{align*}
A \xi_x + \left(B + \sqrt{B^2 - AC}\right) \xi_y &= 0 \\
A \xi_x + \left(B - \sqrt{B^2 - AC}\right) \xi_y &= 0 \\
A \eta_x + \left(B + \sqrt{B^2 - AC}\right) \eta_y &= 0 \\
A \eta_x + \left(B - \sqrt{B^2 - AC}\right) \eta_y &= 0
\end{align*}

We get totally two equations for \( \xi \) and two equations for \( \eta \) but we need one solution for \( \xi \) and one for \( \eta \). We take

\begin{align*}
A \xi_x + \left(B + \sqrt{B^2 - AC}\right) \xi_y &= 0 \\
A \eta_x + \left(B - \sqrt{B^2 - AC}\right) \eta_y &= 0
\end{align*}

The above equations are itself PDE in terms of \( \xi(x, y) \). On the characteristic curves \( \xi(x, y) = \text{constant} \) we get \( d\xi = 0 \Rightarrow \xi_x\, dx + \xi_y\, dy = 0 \). Similarly \( d\eta = 0 \Rightarrow \eta_x\, dx + \eta_y\, dy = 0 \). Equating the like terms gives

\begin{align*}
\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} &= \frac{B + \sqrt{B^2 - AC}}{A} \\
\frac{dy}{dx} = -\frac{\eta_x}{\eta_y} &= \frac{B - \sqrt{B^2 - AC}}{A}
\end{align*}

Solving the above two ordinary differential equations we get two curves \( \xi(x, y) = C_1 \) and \( \eta(x, y) = C_2 \). Thus we get the coordinate transformations by which we can reduce the PDE’S to canonical form.

**Example 2.2.1.** Determine the canonical form of \( u_{tt} - c^2 u_{xx} = 0 \)

\( B^2 - AC = c^2 > 0 \) so the eqn is hyperbolic.
We get
\[ \frac{dx}{dt} = c \text{ and } \frac{dx}{dt} = -c \]
This gives \( \xi(x, y) = x - ct \) and \( \eta(x, y) = x + ct \). Using this we get the canonical form \( w_{\xi \eta} = 0 \Rightarrow w(\xi, \eta) = f(\xi) + g(\eta) \).

So \( u(x, y) = f(x - ct) + g(x + ct) \).

### 2.2.2 Canonical form of parabolic PDE

In order to make \( b^2 - ac = 0 \) we need either \( b = c = 0 \) or \( b = a = 0 \). Let us take \( c = 0 \) because \( b = 0 \) will be automatically forced by parabolic condition.

Let us assume \( A \neq 0 \). This gives
\[ A \eta_x^2 + 2B \eta_x \eta_y + C \eta_y^2 = 0 \]
Substituting \( C = \frac{B^2}{A} \) and factorizing we get
\[ A \eta_x^2 + 2B \eta_x \eta_y + \frac{B^2}{A} \eta_y^2 = 0 \]
\[ \frac{1}{A} (A \eta_x + B \eta_y)^2 = 0 \]
\[ A \eta_x + B \eta_y = 0 \ (\because A \neq 0) \]

On the characteristic curve \( \eta = \text{const} \) we have \( d\eta = 0 \Rightarrow \eta_x \, dx + \eta_y \, dy = 0 \). This gives the transformation \( \eta(x, y) \) from \( -\frac{\eta_x}{\eta_y} = \frac{dy}{dx} = \frac{B}{A} \).

Now we have the freedom to choose the transformation \( \xi(x, y) \) such that \( \xi_x \eta_y - \xi_y \eta_x \neq 0 \). Thus we can reduce the PDE to its canonical form.
Example 2.2.2. Determine the canonical form of \( x^2 u_{xx} - 2xy u_{xy} + y^2 u_{yy} + xu_x + y u_y = 0. \)

\[ B^2 - AC = 0 \] so the eqn is parabolic.

We get

\[ \frac{dy}{dx} = -\frac{y}{x} \Rightarrow xy = \text{const} \]

This implies \( \eta(x, y) = xy. \) Now let us take \( \xi(x, y) = x \) so that \( \xi_x \eta_y - \xi_y \eta_x = x \neq 0. \) You can choose any \( \xi \) satisfying above condition.

The canonical form is \( \xi^2 w_{\xi\xi} + \xi w_{\xi} = 0. \) Solving this finally we get

\[ w(\xi, \eta) = \ln(\xi) g(\eta) + f(\eta) \Rightarrow u(x, y) = \ln(x) g(xy) + f(xy). \]

2.2.3 Canonical form of elliptic PDE

For the elliptic case we need to make \( b^2 - ac = 0, \) we can make \( a = c \) and \( b = 0. \) This gives

\[ A \xi_x^2 + 2B \xi_x \xi_y + C \xi_y^2 = A \eta_x^2 + 2B \eta_x \eta_y + C \eta_y^2 \]

\[ A \xi_x \eta_x + B (\xi_x \eta_y + \xi_y \eta_x) + C \xi_y \eta_y = 0 \]

This is in itself coupled PDE in \( \xi(x, y) \) and \( \eta(x, y). \) Let us try to convert these two equations in complex form by taking \( \Phi = \xi + i \eta. \)

Then we can express the above two equations as

\[ A (\xi_x^2 - \eta_x^2) + 2B (\xi_x \xi_y - \eta_x \eta_y) + C (\xi_y^2 - \eta_y^2) = 0 \]

\[ A \xi_x i \eta_x + B (\xi_x i \eta_y + \xi_y i \eta_x) + C \xi_y i \eta_y = 0 \]

In terms of \( \Phi, \)
\[ A \Phi_x^2 + 2B \Phi_x \Phi_y + C \Phi_y^2 = 0, \]

Factorizing gives

\[
\left( A \Phi_x + \left( B + i\sqrt{AC - B^2} \right) \Phi_y \right) \left( A \Phi_x + \left( B - i\sqrt{AC - B^2} \right) \Phi_y \right) = 0
\]

On \( \Phi = \text{constant} \) we have \( d\Phi = 0 \Rightarrow \Phi_x dx + \Phi_y dy = 0 \).

Equating \( -\Phi_x / \Phi_y = dy/dx \) to the factorization above gives

\[
\frac{dy}{dx} = \frac{B + i\sqrt{AC - B^2}}{A} \quad \frac{dy}{dx} = \frac{B - i\sqrt{AC - B^2}}{A}
\]

This gives \( \varphi(x, y) \) and \( \chi(x, y) \) both are complex functions. From this we can extract real valued function \( \xi(x, y) \) and \( \eta(x, y) \). We will try to understand by an example

**Example 2.2.3.** Determine the canonical form of \( u_{xx} + x^2 u_{yy} = 0 \)

\( B^2 - AC = -x^2 < 0 \) so the eqn is elliptic

We have

\[
\frac{dy}{dx} = \frac{ixy}{2}, \quad \frac{dy}{dx} = -\frac{ix}{2}
\]

After solving these two ODES we get

\[
\varphi(x, y) = \frac{x^2}{2} + iy \quad \chi(x, y) = \frac{x^2}{2} - iy
\]

Taking \( \xi = \frac{\varphi + \chi}{2} \) and \( \eta = \frac{\varphi - \chi}{2i} \) gives \( \xi = \frac{x^2}{2} \) and \( \eta = y \). Using this \( \xi \) and \( \eta \) the canonical form reduces to \( w_{\xi\xi} + w_{\eta\eta} = -\frac{w_\xi}{2\xi} \).
Chapter 3

Method of Separation of variables

3.1 Heat equation

Let us consider the heat conduction problem

\[ u_t - k u_{xx} = 0 \quad 0 < x < L, \ t > 0 \]  \hspace{1cm} (3.1.1)

\[ u(0, t) = u(L, t) = 0, \ t \geq 0 \]  \hspace{1cm} (3.1.2)

\[ u(x, 0) = f(x), \ 0 \leq x \leq L \]  \hspace{1cm} (3.1.3)

Eq (3.2) and (3.3) implies \( f(0) = f(L) = 0 \). Let us assume the solution \( u(x, t) = X(x) \ T(t) \). This gives \( XT' = k \ X'' \ T \).

\[ \frac{T'}{kT} = \frac{X''}{X} = -\lambda \] gives

\[ \frac{d^2X}{dx^2} = -\lambda X, \ 0 < x < L, \ \frac{dT}{dt} = -\lambda k T, \ t > 0 \]
First let us solve the ODE for $X(x)$. We need to find the boundary condition for $X(x)$ at $x = 0$ and $x = L$. Using $u(0, t) = 0 \Rightarrow X(0) T(t) = 0 \Rightarrow X(0) = 0$ and $u(L, t) = 0 \Rightarrow X(L) T(t) = 0 \Rightarrow X(L) = 0$.

$$
\frac{d^2 X}{x^2} + \lambda X = 0, \quad 0 < x < L \quad (3.1.4)
$$

$$
X(0) = X(L) = 0 \quad (3.1.5)
$$

This is an eigenvalue problem with eigenvalue $\lambda$ and eigenfunction $X_\lambda$.

$(\lambda < 0)$ then $X(x) = a \cosh(\sqrt{-\lambda} x) + b \sinh(\sqrt{-\lambda} x)$ with $X(0) = X(L) = 0 \Rightarrow X(x) = 0$.

$(\lambda = 0)$ then $X(x) = a + bx$ with $X(0) = X(L) = 0 \Rightarrow X(x) = 0$.

$(\lambda > 0)$ then $X(x) = a \cos(\sqrt{\lambda} x) + b \sin(\sqrt{\lambda} x)$ with $X(0) = X(L) = 0 \Rightarrow a = 0$ and $\sin\sqrt{\lambda} L = 0 \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, 3..$

Solving ODE for $T(t)$ gives $T_n(t) = e^{-k \left(\frac{m\pi}{L}\right)^2 t}, n = 1, 2, 3,...$. For each $\lambda_n$ the product of $X_n(x) \ast T_n(t)$ is a solution so by the superposition principle we get $u(x, t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} e^{-k \left(\frac{m\pi}{L}\right)^2 t}$. In order to determine the constants $A_n$ we need to use the condition $u(x, 0) = f(x)$.

This gives $\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x)$, Multiplying both sides by $\sin\frac{m\pi x}{L}$ and integrating from 0 to $L$ gives $A_m = \frac{2}{L} \int_{0}^{L} \sin\frac{m\pi x}{L} f(x) \, dx, m = 1, 2, 3,...$

Let us solve a problem with some simple $f(x)$.
Example 3.1.1. Consider the problem

\[ u_t - u_{xx} = 0, \ 0 < x < \pi, \ t > 0 \]
\[ u(0, t) = u(\pi, t) = 0, \ t \geq 0 \]
\[ u(x, 0) = \begin{cases} 
  x, & 0 \leq x \leq \frac{\pi}{2} \\
  \pi - x, & \frac{\pi}{2} \leq x \leq \pi 
\end{cases} \]

We get the solution

\[ u(x, t) = \sum_{n=1}^{\infty} A_n \sin(nx) e^{-k n^2 t} \]

Using the condition of \(u(x, 0)\) we can obtain \(A_n = \frac{4}{\pi n^2} \sin \frac{n\pi}{2}\). The final solution is

\[ u(x, t) = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \left((2n-1)x\right) e^{-(2n-1)^2 t} \]

Our solution is an infinite series of functions. So in order for the solution to be well defined we need to discuss the convergence of the solution. We can use Weierstrass M-test to check the convergence of series of functions \(\sum_{n=1}^{\infty} u_n(x, t)\) where we try to bound \(u_n(x, t) < M_n\) then if the series of real numbers \(\sum_{n=1}^{\infty} M_n\) converges then the real series \(\sum_{n=1}^{\infty} u_n(x, t)\) converges uniformly. In general \(\sum_{n=1}^{\infty} \frac{\partial}{\partial x} u_n(x, t) \neq \sum_{n=1}^{\infty} \frac{\partial}{\partial x} u_n(x, t)\) may not be equal so the convergence of \(\sum_{n=1}^{\infty} \frac{\partial u_n}{\partial t}\) and \(\sum_{n=1}^{\infty} \frac{\partial^2 u_n}{\partial x^2}\) should also be verified. I will leave it as an exercise.
3.2 Wave equation

Let us consider the problem

\[ u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < L, \quad t > 0 \]
\[ u_x(0, t) = u_x(L, t) = 0, \quad 0 \leq x \leq L \]
\[ u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 \leq x \leq L \]

Compatibility conditions

\[ f'(0) = f'(L) = g'(0) = g'(L) = 0 \]

Let \( u(x, t) = X(x) T(t) \) which gives \( X'' = c^2 X' T \)

\[ \frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda \]

From this we get two ordinary differential equations

\[ X'' + \lambda X = 0, \quad T'' + \lambda c^2 T = 0 \]

This is an eigenvalue problem. The boundary condition gives \( u_x(0, t) = 0 \Rightarrow X'(0) T(t) = 0 \Rightarrow X'(0) = 0 \). Similarly \( X'(L) = 0 \).

(\( \lambda < 0 \)) then \( X(x) = a \cosh(\sqrt{-\lambda} x) + b \sinh(\sqrt{-\lambda} x) \) with \( X'(0) = X'(L) = 0 \Rightarrow b = 0 \) and \( \lambda = \left( \frac{n\pi}{L} \right)^2 \) which cannot happen since \( \lambda \) is negative.

(\( \lambda = 0 \)) then \( X(x) = a + bx \) with \( X'(0) = X'(L) = 0 \Rightarrow X_0(x) = \text{const} \).

(\( \lambda > 0 \)) then \( X(x) = a \cos(\sqrt{\lambda} x) + b \sin(\sqrt{\lambda} x) \) with \( X'(0) = X'(L) = 0 \Rightarrow b = 0 \) and \( \sin(\sqrt{\lambda} L) = 0 \Rightarrow \lambda = \left( \frac{n\pi}{L} \right)^2, n = 1, 2, 3, \ldots \)

\( X_n(x) = a_n \cos \left( \frac{n\pi x}{L} \right) \).
Solving the equation $T'' + \lambda c^2 T = 0$

$(\lambda > 0)$ then $T_n(t) = \alpha_n \cos(\sqrt{\lambda_n} c^2 t) + \beta_n \sin(\sqrt{\lambda_n} c^2 t), n = 1, 2, 3,\ldots$

$(\lambda = 0)$ then $T'' = 0 \Rightarrow T_0(t) = \alpha_0 + \beta_0 t$.

Finally $u(x, t) = X_0(x) T_0(t) + \sum_{n=1}^{\infty} X_n(x) T_n(t)$

$\Rightarrow u(x, t) = \frac{A_0 + B_0 t}{2} + \sum_{n=1}^{\infty} \left( A_n \cos \left( \frac{c\pi nt}{L} \right) + B_n \sin \left( \frac{c\pi nt}{L} \right) \right) \cos \left( \frac{n \pi x}{L} \right), n = 1, 2, 3,\ldots$

**Example 3.2.1.** Consider the problem

$$u_{tt} - 4 u_{xx} = 0, \quad 0 < x < 1, \quad t > 0$$

$$u_x(0, t) = u_x(1, t) = 0, \quad t \geq 0$$

$$u(x, 0) = \cos^2(\pi x), \quad 0 \leq x \leq 1$$

$$u_t(x, 0) = \sin^2(\pi x) \cos(\pi x), \quad 0 \leq x \leq 1$$

Using the solution of the previous problem

$$u(x, t) = \frac{A_0 + B_0 t}{2} + \sum_{n=1}^{\infty} (A_n \cos(2n \pi t) + B_n \sin(2n \pi t)) \cos(n \pi x)$$

Using the conditions of initial displacement and initial velocity we get,

$$u(x, 0) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n \pi x) = \cos^2 \pi x = \frac{1 + \cos(2 \pi x)}{2}$$
This gives \( A_0 = 1, A_2 = \frac{1}{2}, A_n = 0, \forall n \neq 0, 2 \)

\[
    u_t(x, 0) = \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n (2n \pi) \cos(n \pi x) = \sin^2(\pi x) \cos(\pi x) = \frac{\cos \pi x}{4} - \frac{\cos 3 \pi x}{4}
\]

This gives \( B_1 = \frac{1}{8 \pi}, B_3 = -\frac{1}{24 \pi}, B_n = 0, \forall n \neq 1, 3 \)

\begin{align*}
    u(x, t) &= \frac{1}{2} + \frac{1}{8 \pi} \sin(2 \pi t) \cos(\pi x) + \frac{1}{2} \cos(4 \pi t) \cos(2 \pi x) - \\
               &\quad \frac{1}{24 \pi} \sin(6 \pi t) \cos(3 \pi x).
\end{align*}

**Example 3.2.2.**

\[
    u_{tt} - u_{xx} = \cos(2 \pi x) \cos(2 \pi t), \quad 0 < x < 1, \ t > 0
\]

\[
    u_x(0, t) = u_x(1, t) = 0, \ t \geq 0
\]

\[
    u(x, 0) = \cos^2(\pi x), \quad 0 \leq x \leq 1
\]

\[
    u_t(x, 0) = 2 \cos(\pi x), \quad 0 \leq x \leq 1
\]

We have \( X_n(x) = \cos(n \pi x), \ \lambda_n = (n \pi)^2, \ n = 0, 1, 2, \ldots \) Let us assume the solution as

\[
    u(x, t) = \frac{T_0(t)}{2} + \sum_{n=1}^{\infty} T_n(t) \cos(n \pi x)
\]

Substituting this expression in the PDE and using the initial conditions finally we get the solution

\[
    u(x, t) = \frac{1}{2} + \left( \frac{\cos(2 \pi t)}{2} + \frac{t + 4}{4 \pi} \sin(2 \pi t) \right) \cos(2 \pi x).
\]
3.3 Mathematical justification of the method

The method of separation of variables has limitations and the points to be noted before applying the method are as follows.

(a) In the partial differential equation \( L u = f(x, y) \)

\[
\left( \text{Ex: For wave equation } L = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} \right) \text{ L must be separable.}
\]

\[ L \phi(x, y) \text{ such that } \frac{L(X(x)Y(y))}{\phi(x, y)X(x)Y(y)} = F(x)G(y) \text{ for some } F \text{ and } G \text{ functions of } x \text{ and } y. \]

(Ex: \( L(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} \), substituting \( u(x, y) = X(x)Y(y) \)

\( L(X(x)Y(y)) = X''Y + X'Y' + XY'' \) cannot be written as \( F(x) + G(y) \). So the PDE is not separable.

(b) All initial and boundary conditions must be on lines \( x = \text{const} \) and \( y = \text{const} \). (Ex: Suppose a domain not rectangular with sides parallel to \( x \) and \( y \) axes.)

(c) Linear operator boundary conditions \( x = \text{const} \) involve no partial derivative of \( u \) w.r.t \( x \). Similarly \( y = \text{const} \) involve no partial derivative of \( u \) w.r.t \( y \). (Ex: Suppose at \( x = 0 \) the condition is given as \( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \).)
Chapter 4

First order partial differential equations

4.1 Introduction

In this chapter we will see the methods to solve first order PDES which are Quasilinear.

\[ a(x, y, u) u_x + b(x, y, u) u_y = c(x, y, u) \]  \hspace{1cm} (4.1.1)

Let us start solving this PDE

Example 4.1.1.

\[ u_x = c_0 u + c_1(x, y), \quad u(0, y) = y \]  \hspace{1cm} (4.1.2)

Eventhough this a PDE since there is no term with \( u_y \) this can be looked as a first-order ODE and then solved by method of integrating
Characteristic curves

\[ \frac{dy}{dx} + P(x) y = Q(x) \]

\[ u(x, y) e^{-c_0 x} = \int_0^x c_1(\xi, y) e^{-c_0 \xi} d\xi + T(y). \]

By using the condition \( u(0, y) = y \). We find that \( T(y) = y \). So this problem has a unique solution.

In solving the above problem by fixing \( y = \text{const} \) we see that the PDE gets converted to set of ODE’s on each \( y = \text{const} \) line. So we have solved the set of ODES’s. These are solutions of ODE’s on the line \( y = \text{const} \). We then call the line as characteristic curve and the solution as characteristic solution.

Now it is not always true that a first-order PDE with given condition will have a solution which is unique. Let us see few examples on
Example 4.1.2. Consider the PDE $u_x = c_0 u$ with $u(x,0) = 2x$.

The solution is $u(x,y) = e^{c_0 x} T(y)$ by using the condition we see that $T(0) = 2x e^{-c_0 x}$ which is impossible. So the PDE has no solution.

Example 4.1.3. Consider the same PDE $u_x = c_0 u$ with $u(x,0) = 2e^{c_0 x}$. we see that $T(0) = 2$ which means there are infinitely many functions $T(y)$ satisfying the above condition. So the PDE has infinitely many solutions.

4.2 Method of Characteristics

Consider the quasi-linear first-order partial differential equation

$$a(x, y, u) u_x + b(x, y, u) u_y = c(x, y, u)$$

The initial condition is given on some curve $\Gamma(s) : (x_0(s), y_0(s), u_0(s))$ in the parametric form where $s \in (\alpha, \beta)$. This can be written as $(a, b, c) \cdot (u_x, u_y, -1) = 0$ where $(u_x, u_y, -1)$ is normal to the surface $u(x, y) - u = 0$ on the $(x, y, u)$ plane. Therefore $(a, b, c)$ lies in the tangent plane. So we get the equations

$$\frac{dx}{dt} = a(x(t), y(t), u(t))$$
$$\frac{dy}{dt} = b(x(t), y(t), u(t))$$
$$\frac{du}{dt} = c(x(t), y(t), u(t))$$

We call this set of ODE’s as characteristic equation. Solving this set of ODE’s gives characteristic curves $(x(t, s), y(t, s), u(t, s))$ where
each curve emanates from different points on $\Gamma(s)$ for the given initial condition $x(0, s) = x_0(s), y(0, s) = y_0(s), u(0, s) = u_0(s)$.

**Example 4.2.1.** Solve the PDE $u_x + u_y = 2$, $u(x, 0) = x^2$

The set of characteristic equations are

$$
\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 1, \quad \frac{du}{dt} = 2
$$

where $x(0, s) = s$, $y(0, s) = 0$, $u(0, s) = s^2$. Solving this set we get $x(t, s) = t + s$, $y(t, s) = t$, $u(t, s) = 2t + s^2$. This gives $u(x, y) = 2y + (x - y)^2$. To see the characteristics in the $x - y$ plane we can use the ODE $\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}$ which gives $y = x + c$ for the current problem.
4.2.1 Transversality condition

In order for the PDE to have a unique solution near the initial curve \( \Gamma(s) \) the transformation \( x(t, s), y(t, s) \) has to be invertible. That is the jacobian should be non-zero on the points of initial curve \( \Gamma \).

\[
\begin{vmatrix}
      x_t & y_t \\
      x_s & y_s \\
\end{vmatrix} = \begin{vmatrix} a & b \\
(x_0)_s & (y_0)_s \end{vmatrix} \neq 0
\]

Geometrically this means projection of \( \Gamma \) on the \( x - y \) plane is tangent at this point to the projection of the characteristic.

Example 4.2.2. Solve the equation \( u_x = 1 \) subject to \( u(0, y) = g(y) \).

\[
\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 0, \quad \frac{du}{dt} = 1
\]

where \( x(0, s) = 0, y(0, s) = s, u(0, s) = g(s) \). Solving this set we get \( x(t, s) = t, y(t, s) = s, u(t, s) = t + g(s) \). This gives \( u(x, y) = x + g(y) \). If the initial condition is changed to \( u(x, 0) = h(x) \), we get where \( x(0, s) = s, y(0, s) = 0, u(0, s) = h(s) \). Solving for this condition we get \( x(t, s) = t + s, y(t, s) = 0, u(t, s) = t + h(s) \). The transversality condition \[
\begin{vmatrix}
1 & 0 \\
1 & 0 \\
\end{vmatrix} = 0
\]

The solution has a problem with uniqueness. The characteristic curves for this equation is \( \frac{dy}{dx} = 0 \Rightarrow y = c \). The initial curve \( \Gamma(s) \) is the \( x \) axis which is also the characteristic curve on \( y = 0 \). So the initial curve \( \Gamma \) and the characteristic curve coincides. So we loose the uniqueness near the initial curve.

Example 4.2.3. Solve the equation \( u_x + u_y + u = 1 \) subject to \( u = \sin x \), on \( y = x + x^2, x > 0 \).

\[
\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 1, \quad \frac{du}{dt} = 1 - u
\]
where $x(0, s) = s$, $y(0, s) = s + s^2$, $u(0, s) = \sin s$. Solving this set we get $x(t, s) = t + s$, $y(t, s) = t + s + s^2$, $u(t, s) = 1 + (\sin s - 1) e^{-t}$.

This gives

$$u(x, y) = 1 + (\sin \sqrt{y - x} - 1) e^{-(x - \sqrt{y - x})}.$$  This solution is valid in the domain $D = \{(x, y) | 0 < x < y\} \cup \{(x, y) | x \leq 0 \text{ and } y = x + x^2\}$.

The transversality condition $2s \neq 0$ when $s \neq 0$. If you observe clearly the solution $u$ is not differentiable at the origin. The characteristics are $\frac{dy}{dx} = 1 \Rightarrow y = x + c$. We can observe that the Slope of the initial curve at the origin = Slope of the characteristic at the origin. When choosing $s$ we have omitted $s = -\sqrt{y - x}$. So the solution near the curve in the region $\{(x, y) | x < 0 \text{ and } y = x + x^2\}$ gives non-uniqueness of the solution.

**Example 4.2.4.** Solve the equation $-yu_x + xu_y = u$ subject to
$u(x,0) = \psi(x)$.
\[
\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x, \quad \frac{du}{dt} = u
\]

where $x(0,s) = s$, $y(0,s) = 0$, $u(0,s) = \psi(s)$. Solving this set we get $x(t,s) = s \cos t$, $y(t,s) = s \sin t$, $u(t,s) = e^t \psi(s)$. This gives
\[
u(x,y) = \psi\left( \sqrt{x^2 + y^2} \right) e^{\tan^{-1}(\frac{x}{y})}.
\]

The transversality condition \[
\begin{vmatrix}
1 & s \\
1 & 0
\end{vmatrix} = -s \neq 0 \text{ when } s \neq 0.
\]
In choosing $\psi\left( \sqrt{x^2 + y^2} \right)$ we have assumed $x > 0$. Each characteristic intersects the initial curve $\Gamma$ twice.

**Example 4.2.5.** Solve the equation $u_x + 3y^2 u_y = 2$ subject to
\[ u(x, 1) = 1 + x. \]

\[
\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 3y^2, \quad \frac{du}{dt} = 2
\]

where \( x(0, s) = s, \ y(0, s) = 1, \ u(0, s) = 1 + s. \) Solving this set we get \( x(t, s) = t + s, \ y(t, s) = (t + 1)^3 \) \( u(t, s) = 2t + s + 1. \) This gives \( u(x, y) = x + y^{\frac{1}{3}}. \) The transversality condition \[
\begin{vmatrix}
1 & 3 \\
1 & 0
\end{vmatrix} = -3 
eq \]

0. The characteristic equation \( \frac{dy}{dt} = 3y^2 \) with \( y(0, s) = 1 \) is not Lipchitz continuous at the origin. This gives \( y = t^3 \) and \( y = 0 \) as solutions. So this does not have unique solution. Hence \( y = 0 \) is also a characteristic. We also get \( y = (x - s + 1)^3. \) For each point \( s \) on \( y = 1 \) we get different characteristic curve which intersects with
another characteristic \( y = 0 \). Thus \( u \) may not have unique solution there. Hence the solution \( u(x, y) = x + y^{\frac{1}{3}} \) is singular on the \( x \)-axis.

### 4.3 Lagrange's Method

Consider the quasilinear PDE 

\[
a(x, y, u) u_x + b(x, y, u) u_y = c(x, y, u).
\]

The characteristic equations are

\[
\frac{dx}{dt} = a(x, y, u), \quad \frac{dy}{dt} = b(x, y, u), \quad \frac{du}{dt} = c(x, y, u)
\]

From this we can extract two set of equations

\[
\frac{dy}{dx} = \frac{b(x, y, u)}{a(x, y, u)}, \quad \frac{du}{dx} = \frac{c(x, y, u)}{a(x, y, u)}
\]

We get two family of curves \( \phi(x, y, u) = \alpha, \psi(x, y, u) = \beta \). So the general solution to the quasilinear PDE is

\[
F(\phi(x, y, u), \psi(x, y, u)) = 0.
\]

**Lemma 4.3.1.** Let \( \phi = \phi(x, y, u), \psi = \psi(x, y, u) \).

If \( f(\phi, \psi) = 0 \) then \( u \) satisfies

\[
\frac{\partial u}{\partial x} \frac{\partial (\phi, \psi)}{\partial (y, u)} + \frac{\partial u}{\partial y} \frac{\partial (\phi, \psi)}{\partial (u, x)} = \frac{\partial (\phi, \psi)}{\partial (x, y)}
\]

**Proof.** We know that \( f(\phi, \psi) = 0 \). Differentiating with respect to \( x \) and \( y \) we get

\[
\frac{\partial f}{\partial \phi} \left( \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x} \right) + \frac{\partial f}{\partial \psi} \left( \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial u} \frac{\partial u}{\partial x} \right) = 0
\]

\[
\frac{\partial f}{\partial \phi} \left( \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial y} \right) + \frac{\partial f}{\partial \psi} \left( \frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial u} \frac{\partial u}{\partial y} \right) = 0
\]
which gives
\[
\frac{\partial u}{\partial x} \frac{\partial (\phi, \psi)}{\partial (y, u)} + \frac{\partial u}{\partial y} \frac{\partial (\phi, \psi)}{\partial (u, x)} = \frac{\partial (\phi, \psi)}{\partial (x, y)}
\]

\[\square\]

**Theorem 4.3.2.** The general solution of PDE \(a(x, y, u) \frac{\partial x}{\partial x} + b(x, y, u) \frac{\partial y}{\partial y} = c(x, y, u)\) is \(f(\phi, \psi) = 0\), where \(\phi(x, y, u) = c_1\), \(\psi(x, y, u) = c_2\) are the solution curves of
\[
\frac{dx}{a(x, y, u)} = \frac{dy}{b(x, y, u)} = \frac{du}{c(x, y, u)}
\]

**Proof.** Since \(\phi(x, y, u) = c_1\) and \(\psi(x, y, u) = c_2\)
\[
d\phi = \phi_x \, dx + \phi_y \, dy + \phi_u \, du = 0
\]
\[
d\psi = \psi_x \, dx + \psi_y \, dy + \psi_u \, du = 0
\]

Using
\[
\frac{dx}{a(x, y, u)} = \frac{dy}{b(x, y, u)} = \frac{du}{c(x, y, u)}
\]

\[
a \phi_x + b \phi_y + c \phi_u = 0
\]
\[
a \psi_x + b \psi_y + c \psi_u = 0
\]

Solving for \(a, b, c\) we get
\[
\frac{a}{\partial (\phi, \psi)} = \frac{b}{\partial (\phi, \psi)} = \frac{c}{\partial (\phi, \psi)}
\]

The proof can be deduced by using the above lemma. \(\square\)
Example 4.3.1. Find the general solution of \( xu_x + y u_y = u \). The set of equations are\[
\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}.
\]
We get \( \phi(x, y, u) = \frac{y}{x} = c_1 \), \( \psi(x, y, u) = \frac{u}{x} = c_2 \). Thus the general solution is \( f\left(\frac{y}{x}, \frac{u}{x}\right) = 0 \). This can be written as \( u(x, y) = x g\left(\frac{y}{x}\right) \).

Example 4.3.2. Find the general solution of \( x^2 u_x + y^2 u_y = (x+y) u \).
The set of equations are
\[
\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{du}{(x + y) u}.
\]
\( \frac{dx}{x^2} = \frac{dy}{y^2} \) gives \( \frac{y-x}{xy} = c_1 \) and \( \frac{dx - dy}{x^2 - y^2} = \frac{du}{(x + y) u} \Rightarrow \frac{x-y}{u} = c_2 \).
The general solution of the above PDE is \( f\left(\frac{y-x}{xy}, \frac{x-y}{u}\right) = 0 \).

Example 4.3.3. Find the general solution of PDE \( u(x+y) + u(x-y) u_y = x^2 + y^2 \), \( u = 0 \) on \( y = 2x \). The set of equations are
\[
\frac{dx}{u(x+y)} = \frac{dy}{u(x-y)} = \frac{du}{x^2 + y^2}.
\]
\( y dx + x dy - u du = x dx - y dy - u du = 0 \) gives \( u^2 - x^2 + y^2 = c_1 \) and \( 2 xy - u^2 = c_2 \). The general solution is \( f\left(x^2 + y^2 - u^2, 2 xy - u^2\right) = 0 \). Using the initial condition \( u = 0 \) on \( y = 2x \). We get \( 4c_1 = 3c_2 \) \( \Rightarrow \)
\[7 u^2 = 6 xy + 4 (x^2 - y^2)\].
Chapter 5

Tutorial 1

Exercise 5.0.1.

\[ \frac{\partial^2 u}{\partial t^2} - 9 \frac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < \infty, \quad t > 0 \]

\[ u(x, 0) = \begin{cases} 1, & \text{if } |x| \leq 2 \\ 0, & \text{if } |x| > 2 \end{cases} \]

\[ \frac{\partial u}{\partial t}(x, 0) = \begin{cases} 1, & \text{if } |x| \leq 2 \\ 0, & \text{if } |x| > 2 \end{cases} \] (5.0.1)

(i) Find \( u(0, \frac{1}{6}) \)?

(ii) Find large time behaviour of the solution \( \lim_{t \to \infty} u(\xi, t) \)? for fixed \( \xi \in \mathbb{R} \).

Exercise 5.0.2. Using D’Alembert’s form of solution for wave equa-
tion find the solution of

\[
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < \infty, \quad t > 0
\]

\[
u(x, 0) = e^{-x^2}, \quad -\infty < x < \infty
\]

\[
\frac{\partial u}{\partial t}(x, 0) = 0, \quad -\infty < x < \infty
\]  

(5.0.2)

**Exercise 5.0.3.** Thermal conductivity of an aluminium alloy is 1.5 W/cm K. Calculate the steady state temperature of an aluminium bar of length 1m (insulated along the sides) with its left end fixed at 20°C and right end fixed at 30°C.

**Exercise 5.0.4.** (a) Show that the function

\[
u(x, t) = e^{-\frac{k \theta^2 x^2}{\rho c}} \sin(\theta x)
\]

is a solution to the homogeneous heat equation \(\rho c \frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < l, \quad \forall t.\)

(b) What values of \(\theta\) will cause \(u\) to also satisfy homogeneous Dirichlet conditions at \(x = 0\) and \(x = l\)?

**Exercise 5.0.5.** Classify the PDE and reduce to its canonical form

(a) \[
\frac{\partial^2 u}{\partial x^2} - 2 \sin x \frac{\partial^2 u}{\partial x \partial y} - \cos^2 x \frac{\partial^2 u}{\partial y^2} - \cos x \frac{\partial u}{\partial y} = 0
\]

(b) \[
y^5 \frac{\partial^2 u}{\partial x^2} - y \frac{\partial^2 u}{\partial y^2} + 2 \frac{\partial u}{\partial y} = 0, \quad y > 0
\]

(c) \[
\frac{\partial^2 u}{\partial x^2} + (1 + y^2) \frac{\partial^2 u}{\partial y^2} - 2y (1 + y^2) \frac{\partial u}{\partial y} = 0
\]
Exercise 5.0.6. Consider the equation
\[ x \frac{\partial^2 u}{\partial x^2} - y \frac{\partial^2 u}{\partial y^2} + \frac{1}{2} \left( \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right) = 0 \]

(a) Find the domain where the equation is elliptic and the domain where it is hyperbolic.

(b) For each of the above two domains, find the corresponding canonical transformation.

Exercise 5.0.7. Using Separation of variables find the solution to the following PDE’s

(a)
\[ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \ t > 0 \]
\[ u(0, t) = u(\pi, t) = 0, \ t \geq 0 \]
\[ u(x, 0) = \sin^3 x, \ 0 \leq x \leq \pi \]
\[ \frac{\partial u}{\partial t}(x, 0) = \sin 2x, \ 0 \leq x \leq \pi \]

(b) Solve the heat equation
\[ \frac{\partial u}{\partial t} = 12 \frac{\partial^2 u}{\partial x^2}, \ 0 < x < \pi, \ t > 0 \]
subject to the following boundary and initial condition
\[ \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(\pi, t) = 0, \ t \geq 0 \]
\[ u(x, 0) = 1 + \sin^3 x, \ 0 \leq x \leq \pi \]

Find \( \lim_{t \to \infty} u(x, t) \) for all \( 0 < x < \pi \) and explain physical interpretation of your result.
(c) Consider the heat equation \( \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \), \( x \in \mathbb{R}, \ t \geq 0 \). Find the transformation \( \lambda(x, t) \) by solving an ODE of the form \( \phi(\lambda) \).
Chapter 6

Tutorial II

Exercise 6.0.8. Solve the PDE subject to the given Cauchy condition on the Cauchy data curve

(1) \(2 u_x - 5 u_y = 4\), subject to the Cauchy condition \(u(x, 0) = x\).

(2) \(u_x + 3 u_y = u + 2\), subject to the Cauchy condition \(u(0, y) = y\).

(3) \(x u_x + 2 y u_y = 3 u\), subject to the Cauchy condition \(u(1, y) = \cos y\) for \(y > 1\).

(4) \(u_x - 2 u_y = u - 1\), subject to the Cauchy condition \(u = 2y\) on \(x = ky\), giving reasons for any restriction that must be placed on \(k\).

(5) Solve by the method of characteristic \(u_x + 2 x u_y = 2 x u\), given that \(u = x^2\) on the initial curve \(\Gamma\) with the equation \(y = \frac{x^2}{2}\).

(6) \(u_t + 2 u u_x = 0\), subject to the Cauchy condition \(u(x, 0) = \tanh x\).
(7) $u_t - uu_x = e^t$, subject to the Cauchy condition $u(x, 0) = -x$.

(8) $uu_t + uu_x = 0$, subject to the Cauchy condition $u(x, 1) = \frac{1}{x}$ for $x \geq 1$.

(9) $u_t + uu_x = t$, subject to the Cauchy condition $u(x, 0) = -2x$.

Find the domain in the upper half of $(x, t)$ plane where the solution is valid.

(10) Solve the PDE $u_t + 3u^3u_x = 0$ subject to the Cauchy condition

$$u(x, 0) = \begin{cases} -1, & -\infty < x < -1 \\ x, & -1 \leq x \leq 4 \\ 4, & 4 < x < \infty \end{cases}$$

**Exercise 6.0.9.** Solve the following PDE by Lagrange’s method

(1) $3u_x + 2u_y = 0$ with $u(x, 0) = \sin(x)$

(2) $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$

(3) $yu_x + xu_y = xy$, $x \geq 0, y \geq 0$ with $u(0, y) = e^{-y^2}$

for $y > 0$, $u(x, 0) = e^{-x^2}$, for $x > 0$

(4) $2xyu_x + (x^2 + y^2)u_y = 0$, $u = e^\frac{x}{x-y}$ on $x + y = 1$.