Dynamic Boundary Condition at the Fluid Interface

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This short note outlines the derivation of dynamic boundary condition at the interface between two immiscible fluids.

Associated with a surface of area A between two fluids, there is a free energy per unit area which we denote by σ. If the surface area changes by an amount dA, then there is a corresponding change in the energy, σdA. This is equivalent to considering that the interface behaves like a stretched elastic membrane under tension which has a natural tendency to minimize their surface area. This tension force per unit length tangential to the surface may be interpreted as surface tension. The free energy per unit area may also be interpreted as the surface tension. Thus, the surface tension has two, seemingly different, interpretations. We now derive an expression that relates the jump in the stress across the interface and the surface tension.

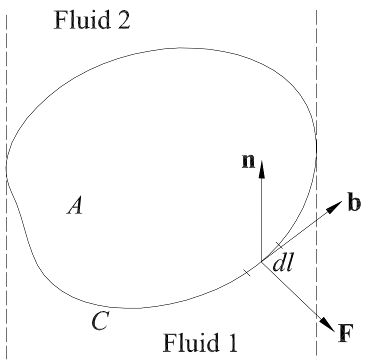

Figure 1: A fluid segment at the interface.

Consider a segment of fluid interface with surface area, A, as shown in figure 1. The boundary of this segment is denoted by C. Let $\mathbf{n}$ be the unit normal to the surface, pointing from fluid 1 to fluid 2 and $\mathbf{b}$ be the unit tangent to C. The magnitude of the surface tension force acting over the elemental length $d\ell$ is given by, $dF_{st} = \sigma d\ell$. This force act in a direction normal to both $\mathbf{n}$ and $\mathbf{b}$. Thus the surface tension vector can be represented as

$$d\boldsymbol{F}_{st} = \sigma d\ell \mathbf{b} \times \mathbf{n}.$$

For the sake convenience in further manipulation, we rewrite the above equation using Cartesian tensor notation:

$$dF_{st_i} = \sigma \varepsilon_{ijk} b_j n_k d\ell \tag{1}$$

where ε_{ijk} is the *Levy-Civita operator*, defined by

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } i,j,k \text{ are in cyclic order and distinct;} \\ -1 & \text{if } i,j,k \text{ are in noncyclic order and distinct;} \\ 0 & \text{if any of } i,j,k \text{ repeats.} \end{cases}$$

Now to calculate the total surface tension force, we integrate equation (1) along the boundary C. Thus,

$$F_{st_i} = \oint_C \sigma \varepsilon_{ijk} b_j n_k d\ell.$$

Define $a_j = \sigma \varepsilon_{ijk} n_k$, so that

$$F_{st_i} = \oint_C a_j b_j d\ell = \oint_C \mathbf{a} \cdot \mathbf{b}\, d\ell = \oint_C \mathbf{a} \cdot d\boldsymbol{\ell}.$$

Using Stokes' theorem,

$$\oint_C \mathbf{a} \cdot d\boldsymbol{\ell} = \int_S (\nabla \times \mathbf{a}) \cdot \mathbf{n} dS$$

which can be written in Cartesian tensor notation as,

$$\oint_C a_j b_j d\ell = \int_S (\nabla \times \mathbf{a})_m n_m dS = \int_S \varepsilon_{mpq} \frac{\partial a_q}{\partial x_p} n_m dS.$$

Thus, the surface tension force is given by

$$\begin{aligned} F_{st_i} &= \int_S \varepsilon_{mpq} \frac{\partial a_q}{\partial x_p} n_m dS \\ &= \int_S \varepsilon_{mpq} \frac{\partial}{\partial x_p} (\sigma \varepsilon_{iqk} n_k) n_m dS \\ &= \int_S \varepsilon_{mpq}\, \varepsilon_{iqk} \frac{\partial}{\partial x_p} (\sigma n_k) n_m dS \\ &= -\int_S \varepsilon_{mpq}\, \varepsilon_{ikq} \left(\frac{\partial \sigma}{\partial x_p} n_k + \sigma \frac{\partial n_k}{\partial x_p} \right) n_m dS. \end{aligned} \tag{2}$$

We now use the following identity (Aris [1]):

$$\varepsilon_{mpq}\, \varepsilon_{ikq} = \delta_{mi} \delta_{pk} - \delta_{mk} \delta_{pi}$$

where δ_{ij} is the *Kronecker delta*, defined by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

Introducing the above identity in equation (2), we have

$$\begin{aligned} F_{st_i} &= \int_S \delta_{mk}\, \delta_{pi} \left(\frac{\partial \sigma}{\partial x_p} n_k + \sigma \frac{\partial n_k}{\partial x_p} \right) n_m dS - \int_S \delta_{mi}\, \delta_{pk} \left(\frac{\partial \sigma}{\partial x_p} n_k + \sigma \frac{\partial n_k}{\partial x_p} \right) n_m dS \\ &= \int_S \left(\frac{\partial \sigma}{\partial x_i} n_k + \sigma \frac{\partial n_k}{\partial x_i} \right) n_k dS - \int_S \left(\frac{\partial \sigma}{\partial x_k} n_k + \sigma \frac{\partial n_k}{\partial x_k} \right) n_i dS. \end{aligned} \tag{3}$$

Since,

$$\begin{aligned}
n_k n_k &= 1, \\
\frac{\partial n_k}{\partial x_k} &= \nabla \cdot \mathbf{n}, \\
\frac{\partial n_k}{\partial x_i} n_k &= \frac{\partial}{\partial x_i}\left(\frac{n_k n_k}{2}\right) = 0,
\end{aligned}$$

equation (3) can be written as

$$F_{st_i} = \int_S \left[\frac{\partial \sigma}{\partial x_i} - \left(\frac{\partial \sigma}{\partial x_k} n_k\right) n_i\right] dS - \int_S \sigma (\nabla \cdot \mathbf{n}) n_i dS. \tag{4}$$

Equation (4) gives the i^{th} component of the surface tension force. Reverting to the vector notation, the surface tension force vector is given by

$$\begin{aligned}
\boldsymbol{F}_{st} &= \int_S \left[\nabla \sigma - (\mathbf{n} \cdot \nabla \sigma)\mathbf{n}\right] dS - \int_S \sigma (\nabla \cdot \mathbf{n}) \mathbf{n} dS \\
&= \int_S \nabla \sigma dS - \int_S \sigma \kappa \mathbf{n} dS
\end{aligned} \tag{5}$$

where in the last step, $\mathbf{n} \cdot \nabla \sigma = \partial \sigma / \partial n$ is set to zero, since σ being a surface property, does not vary with respect to $\mathbf{n}$. κ is the mean curvature of the interface given by

$$\kappa = \nabla \cdot \mathbf{n} = \frac{1}{R_1} + \frac{1}{R_2}$$

where R_1 and R_2 are the principle radii of curvature of the interface in any two orthogonal planes containing $\mathbf{n}$, being reckoned here as *positive* when the corresponding center of curvature lies on the opposite side of the interface to which $\mathbf{n}$ points. In the limit, as area $A \to 0$, the equation (5) becomes

$$\lim_{A \to 0} \frac{\boldsymbol{F}_{st}}{A} = \nabla \sigma - \sigma \kappa \mathbf{n}. \tag{6}$$

Now consider an elemental control volume with a thickness h around the surface S as shown

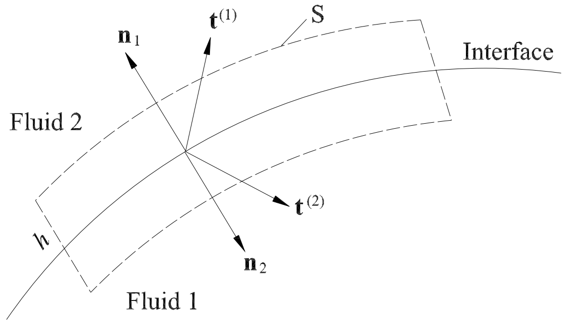

Figure 2: Control volume enclosing the interface.

in figure 2. The forces that act on the surface of this control volume are the surface traction $\mathbf{t}^{(1)}$

and $\mathbf{t}^{(2)}$ which act on either side of the control volume surface, and the surface tension force σ which acts along the direction of vector $\mathbf{b} \times \mathbf{n}$. By Newton's second law, the mass $\times$ acceleration of the control volume is equal to the net force acting on the control volume, i.e.,

$$\begin{aligned}\rho \mathscr{V} \mathbf{a} &= \Sigma \boldsymbol{F} \\ &= (\mathbf{t}^{(1)} + \mathbf{t}^{(2)})A + \boldsymbol{F}_{st} + \rho \mathscr{V} \mathbf{g}\end{aligned}$$

where $\mathscr{V} = Ah$ is the volume of the control volume. Using Cauchy's relation (see Batchelor [2]), the traction vector can be related to the stress tensor, $\boldsymbol{T}$. Thus, $\mathbf{t}^{(1)} = \boldsymbol{T}^{(1)} \cdot \mathbf{n}_1$ and $\mathbf{t}^{(2)} = \boldsymbol{T}^{(2)} \cdot \mathbf{n}_2$. Now, the force balance equation can be written as

$$\boldsymbol{T}^{(1)} \cdot \mathbf{n}_1 + \boldsymbol{T}^{(2)} \cdot \mathbf{n}_2 + \frac{\boldsymbol{F}_{st}}{A} = \frac{\rho \mathscr{V}}{A}(\mathbf{a} - \mathbf{g}).$$

In the limit, as the thickness $h \to 0$ and the area $A \to 0$, the right-hand side of the above equation goes to zero. Thus, we have

$$\lim_{A \to 0} \frac{\boldsymbol{F}_{st}}{A} = \boldsymbol{T}^{(2)} \cdot \mathbf{n}_1 - \boldsymbol{T}^{(1)} \cdot \mathbf{n}_1 \qquad \text{(since } \mathbf{n}_2 = -\mathbf{n}_1\text{)}. \tag{7}$$

Letting $\mathbf{n}_1 = \mathbf{n}$ (i.e., $\mathbf{n}$ is assumed to points from fluid 1 to fluid 2), and using equations (6) and (7), we get the following relation

$$\left[\boldsymbol{T}^{(2)} - \boldsymbol{T}^{(1)}\right] \cdot \mathbf{n} = \nabla\sigma - \sigma\kappa\mathbf{n}. \tag{8}$$

This is the general form of the *dynamic boundary condition* at the interface, which gives the fundamental relationship between the *jump* in stress across an interface and the surface tension force. As pointed out by Landau and Lifshitz [3], this condition can be satisfied only for a viscous fluid. The i^{th} component of the equation may be written as

$$\tau_{ij}^{(2)} n_j - \tau_{ij}^{(1)} n_j = \frac{\partial \sigma}{\partial x_i} - \sigma\kappa n_i.$$

Taking the dot product of both sides of equation (8) with $\mathbf{n}$, setting $\mathbf{n} \cdot \nabla\sigma = 0$, we get the jump in normal stress as

$$[\tau_{nn}] = \tau_{nn}^{(2)} - \tau_{nn}^{(1)} = -\sigma\kappa. \tag{9}$$

For fluid under static equilibrium, equation (9) reduces to the classic *Young-Laplace equation*

$$[p] = p^{(2)} - p^{(1)} = \sigma\kappa, \tag{10}$$

which states that the pressure jump across the interface is balanced by the interfacial surface tension.

The jump in shear stress is obtained analogously by taking the dot product of both sides of the equation (8) with $\mathbf{s}$, where $\mathbf{s}$ is one of the unit tangents to surface S; we get

$$[\tau_{ns}] = \tau_{ns}^{(2)} - \tau_{ns}^{(1)} = \frac{\partial \sigma}{\partial s}. \tag{11}$$

Note that, if the surface tension coefficient σ does not vary along the interface, the jump in shear stress is zero and hence, the shear stress is continuous across the interface. Surface tension can be assumed to be uniform, in the absence of temperature variation or gradient of any surfactant along the interface.

References

[1] Aris, R., *Vectors, Tensors, and the Basic Equations of Fluid Mechanics*, Dover Publications (1990).

[2] Batchelor, G. K., *An Introduction to Fluid Dynamics*, Cambridge University Press (2000).

[3] Landau, L. D. and Lifshitz, E. M., *Fluid Mechanics*, 2^{nd} ed., Pergamon Press (1987).