Finite Difference Method for Parabolic PDE

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1 Introduction

We consider the time-dependent diffusion equation describing a damped diffusion in time

$$u_t = \alpha u_{xx} \tag{1}$$

The initial and Dirichlet type boundary conditions are given as follows:

$$\begin{aligned}
\text{At } t=0 \qquad & u(x,0) = u_0(x) \qquad 0 \le x \le L \\
\text{At } x=0 \qquad & u(0,t) = g_a(t) \qquad t \ge 0 \\
\text{At } x=L \qquad & u(L,t) = g_b(t) \qquad t \ge 0
\end{aligned} \tag{2}$$

An analytical solution for the equation (1) defined in the domain $0 \le x \le \pi$ can be obtained for the following initial and boundary conditions:

Boundary conditions:

$$u(0,t) = u_a \qquad u(\pi,t) = u_b \tag{3}$$

Initial condition:

$$u(x,0) = \sum_{m=1}^{M} \hat{u}_m(0)\sin k_m x + h(x) \tag{4}$$

where

$$h(x) = u_a + (u_b - u_a)\frac{x}{\pi}$$

is the steady-state solution.

For the above conditions the analytical solution of equation (1) is given by

$$u(x,t) = \sum_{m=1}^{M} \hat{u}_m(0)e^{-\alpha k_m^2 t}\sin k_m x + h(x) \tag{5}$$

The following inference can be made from the solution:

- The boundary values u_a and u_b influence the values of $u(x,t)$ at every point in the domain.

- Only initial conditions are required (i.e., conditions at $t = 0$). No final conditions are required, for example conditions at $t \to \infty$. We do not need to know the future to solve this problem!
- The initial conditions influence the values of u at every point in the domain for all future times. The amount of influence decreases with time, and may affect different spatial points to different degrees.
- A steady state is reached for $t \to \infty$. Here, the solution becomes independent of $\sum_{m=1}^{M} \hat{u}_m(0) \sin k_m x$. It also recovers its elliptic spatial behavior.
- The temperature is bounded by its initial and boundary conditions in the absence of source terms.

It is clear from this problem that the variable t behaves very differently from the variable x. The variation in t admits only one-way influences, whereas the variable x admits two-way influences. t is sometimes referred to as the marching or parabolic direction.

2 Basic Numerical Schemes

To facilitate the numerical solution, the one-dimensional domain is discretized with a uniform grid as shown in figure 1 . The first grid point (the one on the left boundary) is labelled point 1. The points are evenly distributed along the x axis, with Δx denoting the spacing between grid points. The last point, namely, that at the right boundary, is denoted by N. Thus, we have a total number of N grid points distributed along the axis.

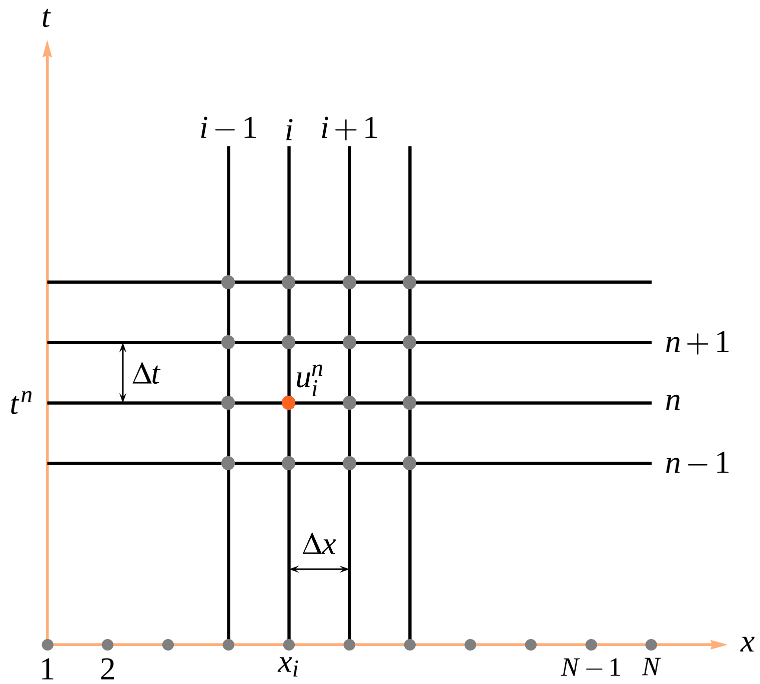

Figure 1: Mesh showing discretization of time and space domain.

FTCS scheme

The semi-discretized form of equation (1) at spatial location i and time level n may be written as

$$(u_t)_i^n = \alpha \left(u_{xx}\right)_i^n \tag{6}$$

Then the explicit FTCS scheme is given by

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \tag{7}$$

or

$$u_i^{n+1} = u_i^n + d\left(u_{i-1}^n - 2u_i^n + u_{i+1}^n\right) \tag{8}$$

where d is the dimensionless diffusion number (or grid Fourier number).

$$d = \frac{\alpha \Delta t}{\Delta x^2}$$

By definition, (8) is explicit because u_i^{n+1} at time step $n+1$ can be solved explicitly in terms of the known quantities at the previous time step n, thus called an explicit scheme.

Order of accuracy of the scheme is $O(\Delta t, \Delta x^2)$. The method is conditionally stable, and the stability condition is given by $d \leq 0.5$.

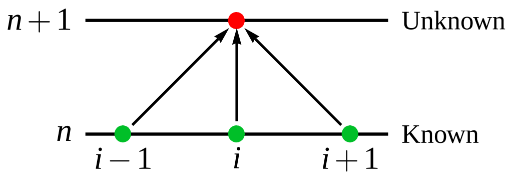

Figure 2: Computational molecule for the explicit FTCS scheme.

BTCS scheme

The semi-discretized form of equation (1) at spatial location i and time level $n+1$ may be written as

$$(u_t)_i^{n+1} = \alpha \left(u_{xx}\right)_i^{n+1} \tag{9}$$

Then the implicit BTCS scheme is given by

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} \tag{10}$$

or

$$du_{i-1}^{n+1} - (1+2d)u_i^{n+1} + du_{i+1}^{n+1} = -u_i^n \tag{11}$$

Writing this equation for all grid points at $n+1$ time level, leads to a tridiagonal system and can be solved using TDMA algorithm. The BTCS scheme is also known as the Laasonen method. This is unconditionally stable. Order of accuracy of the scheme is $O(\Delta t, \Delta x^2)$.

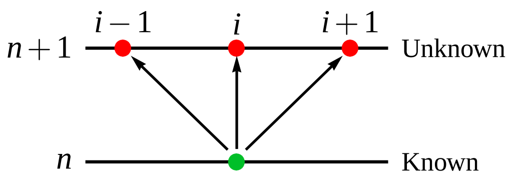

Figure 3: Computational molecule for the implicit BTCS scheme.

Richardson method

Richardson method is a Central Time Central Space (CTCS) scheme for parabolic type diffusion equations. The application of central differencing for time and space derivative in a straightforward manner to equation (1) will yield

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \tag{12}$$

This is known as the Richardson method. Order of accuracy of the scheme is $O(\Delta t^2, \Delta x^2)$. A stability analysis would show that it is unconditionally unstable, no matter how small Δt is. Thus, it is of no practical use. It may be noted that the unstable behavior refers to the equation as a whole. It is *a* stable method for convection equation.

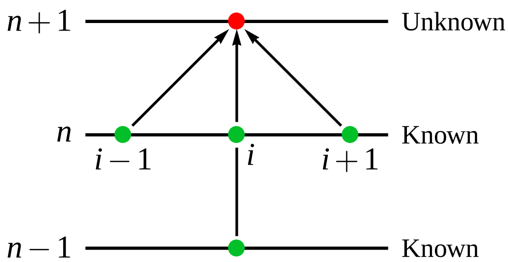

Figure 4: Computational molecule for the Richardson Method.

Dufort–Frankel scheme

The Richardson method can be modified to produce a stable algorithm. This is achieved by replacing u_i^n on the right-hand side with the time-average of previous and current time values at $n-1$ and $n+1$. This new formulation is called *Dufort–Frankel scheme* and is given by

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \frac{\alpha}{\Delta x^2} \left[u_{i+1}^n - 2\left(\frac{u_i^{n-1} + u_i^{n+1}}{2} \right) + u_{i-1}^n \right]$$

or

$$u_i^{n+1} = u_i^{n-1} + \frac{2\alpha\Delta t}{\Delta x^2} \left(u_{i+1}^n - u_i^{n-1} - u_i^{n+1} + u_{i-1}^n \right)$$

or

$$(1+2d)u_i^{n+1} = (1-2d)u_i^{n-1} + 2d\left(u_{i-1}^n + u_{i+1}^n \right) \tag{13}$$

This scheme is explicit and can be shown to be unconditionally stable by the *von Neumann stability analysis*. Since Dufort–Frankel stencil is constructed on the basis of an ad-hoc modification of the Richardson scheme, its consistency must be examined by computing the *modified equation*. Note that the Dufort–Frankel method is a two-level method since the

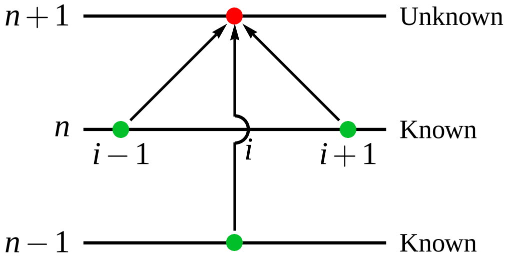

Figure 5: Computational molecule for the Dufort–Frankel scheme.

stencil contains values of u at two time levels other than the current level n. Consequently, to start the computation, values of u at n and $n-1$ are required. Therefore, either two sets of initial data must be available or from a practical point of view, a one-step method may be used as a starter to generate additional data.

Order of accuracy of the scheme is $O(\Delta t^2, \Delta x^2, (\Delta t/\Delta x)^2)$. Even though the method is unconditionally stable, accurate solution will be obtained only if $\Delta t \ll \Delta x$.

Crank–Nicolson scheme

Both FTCS and BTCS schemes are first-order in time and second-order in space. Since they are first-order accurate in time, the time step Δt must be kept small to ensure acceptable accuracy. A scheme having a second-order accuracy in time for parabolic PDE can be

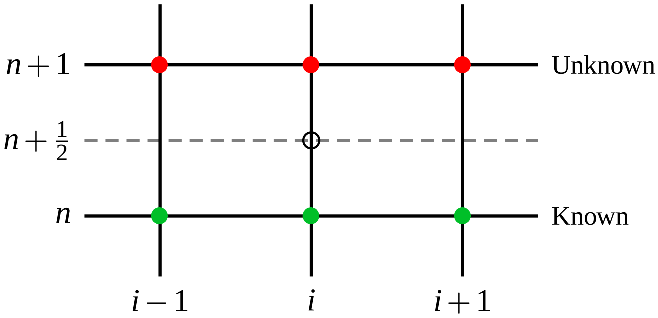

Figure 6: Grid points for the Crank–Nicolson scheme.

obtained by taking the average of the FTCS and BTCS schemes. The new scheme known as the *Crank–Nicolson scheme* [1] or *trapezoidal differencing scheme* named after their inventors John Crank and Phyllis Nicolson. The finite difference approximation of the model equation at $n+1/2$ time level can be written as

$$(u_t)_i^{n+\frac{1}{2}} = \alpha\,(u_{xx})_i^{n+\frac{1}{2}} = \frac{\alpha}{2}\left[(u_{xx})_i^n + (u_{xx})_i^{n+1}\right]$$

where we have expressed u_{xx} at $n+1/2$ time level by the average of the previous and current time values at n and $n+1$ respectively. The time derivative at $n+1/2$ time level and the space derivatives may now be approximated by second-order central difference approximations, yielding

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha}{2}\left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}\right]$$

or

$$u_i^{n+1} = u_i^n + \frac{d}{2}\left[\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) + \left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right)\right] \tag{14}$$

Crank–Nicolson method can also be written as an algorithm

$$0.5du_{i-1}^{n+1} - (1+d)u_i^{n+1} + 0.5du_{i+1}^{n+1} = -u_i^n - 0.5d\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

or

$$du_{i-1}^{n+1} - 2(1+d)u_i^{n+1} + du_{i+1}^{n+1} = -2u_i^n - d\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) \tag{15}$$

A stability analysis would indicate that this implicit method is unconditionally stable.

Generalized implicit method

A general form of the finite difference approximation for diffusion equation may be obtained from Crank–Nicolson method by expressing space derivative by a weighted average of previous and current time values at n and $n+1$. That is

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha\left[(1-\beta)\left(u_{xx}\right)_i^n + \beta\left(u_{xx}\right)_i^{n+1}\right]$$

or

$$u_i^{n+1} = u_i^n + d\left[(1-\beta)\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) + \beta\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right)\right] \tag{16}$$

where in practice $0 < \theta < 1$. This is known as the β-method. $\beta = 0$ gives the explicit FTCS scheme, $\beta = 1$ gives the fully implicit BTCS scheme, and $\beta = 1/2$ gives the Crank–Nicolson method. For $1/2 \le \beta \le 1$, the method is unconditionally stable, but for $0 \le \beta < 1/2$

$$d = \frac{\alpha \Delta t}{\Delta x^2} \le \frac{1}{2(1-2\beta)} \tag{17}$$

3 Schemes for Multi-Dimensional Parabolic PDEs

Let us now examine the solution of the two-dimensional diffusion equation,

$$\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \Rightarrow \qquad u_t = \alpha\left(u_{xx} + u_{yy}\right) \tag{18}$$

with the forward difference in time and the central difference in space (FTCS). We write an explicit scheme in the form

$$\frac{u_{ij}^{n+1} - u_{ij}^n}{\Delta t} = \alpha\left[\frac{u_{i+1,j}^n - 2u_{ij}^n + u_{i-1,j}^n}{\Delta x^2} + \frac{u_{i,j+1}^n - 2u_{ij}^n + u_{i,j-1}^n}{\Delta y^2}\right]$$

which may be written as

$$u_{ij}^{n+1} = u_{ij}^{n} + d_x\left(u_{i+1,j}^{n} - 2u_{ij}^{n} + u_{i-1,j}^{n}\right) + d_y\left(u_{i,j+1}^{n} - 2u_{ij}^{n} + u_{i,j-1}^{n}\right) \tag{19}$$

where

$$d_x \equiv \frac{\alpha\Delta t}{\Delta x^2} \qquad \text{and} \qquad d_y \equiv \frac{\alpha\Delta t}{\Delta y^2} \tag{20}$$

Using stability analysis, it can be shown that the system is stable if

$$d_x + d_y \le \frac{1}{2} \tag{21}$$

For simplicity, let $d_x = d_y = d$ for $\Delta x = \Delta y$. This will give $d \le 1/4$ for stability, which is twice as restrictive. To avoid this restriction, consider the generalized implicit scheme

$$\frac{u_{ij}^{n+1} - u_{ij}^{n}}{\Delta t} = \alpha\left[(1-\beta)\left(u_{xx} + u_{yy}\right)_i^n + \beta\left(u_{xx} + u_{yy}\right)_i^{n+1}\right]$$

The use of central differencing scheme for space derivative yields

$$\begin{aligned}\frac{u_{ij}^{n+1} - u_{ij}^{n}}{\Delta t} = \alpha\Bigg[&(1-\beta)\left(\frac{u_{i+1,j}^{n} - 2u_{ij}^{n} + u_{i-1,j}^{n}}{\Delta x^2} + \frac{u_{i,j+1}^{n} - 2u_{ij}^{n} + u_{i,j-1}^{n}}{\Delta y^2}\right)\\ &+\beta\left(\frac{u_{i+1,j}^{n+1} - 2u_{ij}^{n+1} + u_{i-1,j}^{n+1}}{\Delta x^2} + \frac{u_{i,j+1}^{n+1} - 2u_{ij}^{n+1} + u_{i,j-1}^{n+1}}{\Delta y^2}\right)\Bigg]\end{aligned} \tag{22}$$

or

$$\begin{aligned}&(1 + 2\beta d_x + 2\beta d_y)\,u_{ij}^{n+1} - \beta d_y\,u_{i,j-1}^{n+1} - \beta d_x\,u_{i-1,j}^{n+1} - \beta d_x\,u_{i+1,j}^{n+1} - \beta d_y\,u_{i,j+1}^{n+1}\\ &\qquad = u_{ij}^{n} + (1-\beta)\left[d_x\left(u_{i+1,j}^{n} - 2u_{ij}^{n} + u_{i-1,j}^{n}\right) + d_y\left(u_{i,j+1}^{n} - 2u_{ij}^{n} + u_{i,j-1}^{n}\right)\right]\end{aligned} \tag{23}$$

A variety of implicit schemes can be recovered from generalized implicit, for example, the Crank–Nicolson is obtained by setting $\beta = 1/2$.

$$\begin{aligned}\frac{u_{ij}^{n+1} - u_{ij}^{n}}{\Delta t} = \frac{\alpha}{2}\Bigg[&\left(\frac{u_{i+1,j}^{n} - 2u_{ij}^{n} + u_{i-1,j}^{n}}{\Delta x^2} + \frac{u_{i,j+1}^{n} - 2u_{ij}^{n} + u_{i,j-1}^{n}}{\Delta y^2}\right)\\ &+\left(\frac{u_{i+1,j}^{n+1} - 2u_{ij}^{n+1} + u_{i-1,j}^{n+1}}{\Delta x^2} + \frac{u_{i,j+1}^{n+1} - 2u_{ij}^{n+1} + u_{i,j-1}^{n+1}}{\Delta y^2}\right)\Bigg]\end{aligned} \tag{24}$$

or

$$\begin{aligned}&(1 + d_x + d_y)\,u_{ij}^{n+1} - \frac{1}{2}\left(d_y\,u_{i,j-1}^{n+1} - d_x\,u_{i-1,j}^{n+1} - d_x\,u_{i+1,j}^{n+1} - d_y\,u_{i,j+1}^{n+1}\right)\\ &\qquad = u_{ij}^{n} + \frac{d_x}{2}\left(u_{i+1,j}^{n} - 2u_{ij}^{n} + u_{i-1,j}^{n}\right) + \frac{d_y}{2}\left(u_{i,j+1}^{n} - 2u_{ij}^{n} + u_{i,j-1}^{n}\right)\end{aligned} \tag{25}$$

Alternating Direction Implicit (ADI) method

It is clear that, when implicit schemes are applied to multidimensional problems, the resulting implicit matrix system is not tridiagonal anymore as for three point discretizations on one-dimensional equations. Since each discretized equation consists of five unknowns, we obtain a pentadiagonal matrix system. Unfortunately, we do not have an efficient direct solver, such as Thomas algorithm, for the solution of a pentadiagonal matrix system. However, a multidimensional problem can be split into a series of pseudo-one-dimensional problems and each of which can be solved using Thomas algorithm.

More specifically, in a two-dimensional problem, each time step is split into two sub step of equal duration $\Delta t/2$ and approximating the spatial derivative in a partially implicit manner while alternating between x and y directions. This method is called *Alternating Direction Implicit (ADI) method*. The following are the two steps of ADI method by Peaceman and Rachford [2].

$$\frac{u_{ij}^{n+\frac{1}{2}} - u_{ij}^{n}}{\Delta t/2} = \alpha\left(\frac{u_{i+1,j}^{n+\frac{1}{2}} - 2u_{ij}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}}{\Delta x^2} + \frac{u_{i,j+1}^{n} - 2u_{ij}^{n} + u_{i,j-1}^{n}}{\Delta y^2}\right) \tag{26}$$

$$\frac{u_{ij}^{n+1} - u_{ij}^{n+\frac{1}{2}}}{\Delta t/2} = \alpha\left(\frac{u_{i+1,j}^{n+\frac{1}{2}} - 2u_{ij}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}}{\Delta x^2} + \frac{u_{i,j+1}^{n+1} - 2u_{ij}^{n+1} + u_{i,j-1}^{n+1}}{\Delta y^2}\right) \tag{27}$$

Either equation (26) or (27), as a method in its own right, is only first-order accurate in time

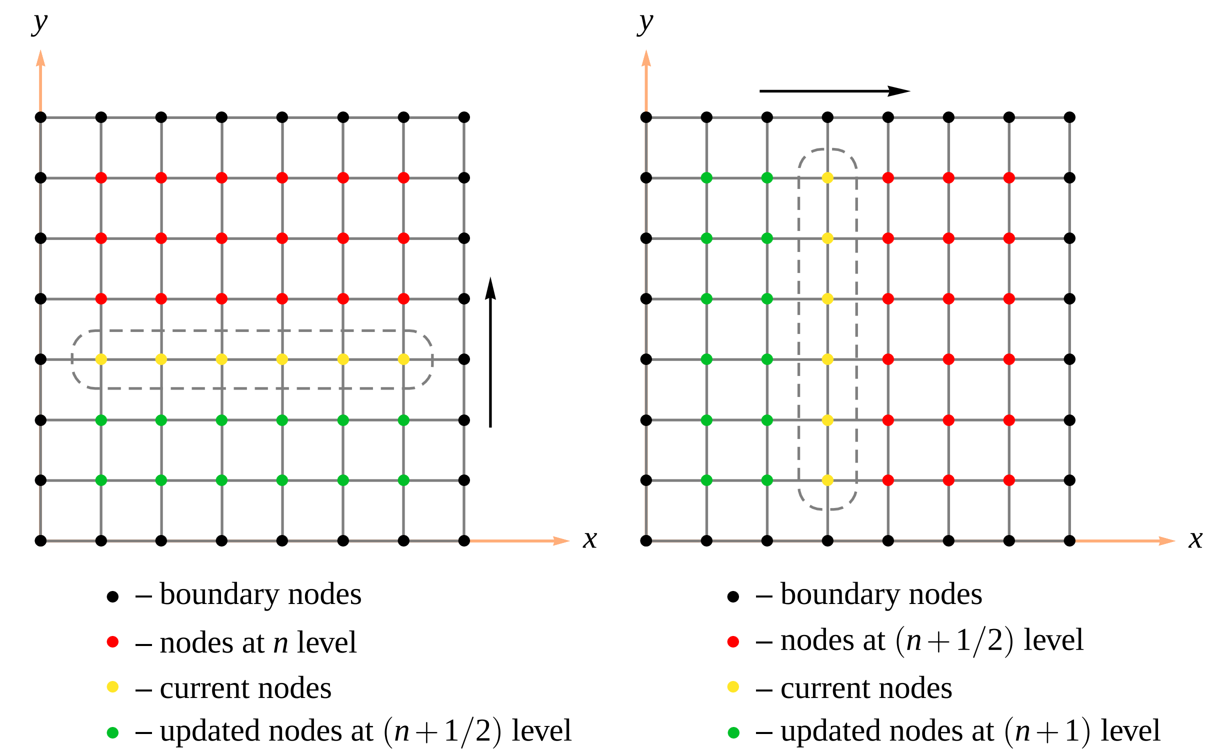

Figure 7: Grid system for ADI method.

and conditionally stable but the combined method is second-order accurate ($O(\Delta t^2, \Delta x^2, \Delta y^2)$) and unconditionally stable! These two equations can be written in a tridiagonal form as follows:

$$-\frac{d_x}{2} u_{i-1,j}^{n+\frac{1}{2}} + (1 + d_x)\, u_{ij}^{n+\frac{1}{2}} - \frac{d_x}{2} u_{i+1,j}^{n+\frac{1}{2}} = u_{i,j}^n + \frac{d_y}{2}\left(u_{i,j-1}^n - 2u_{i,j}^n + u_{i,j+1}^n\right) \tag{28}$$

and

$$-\frac{d_y}{2} u_{i,j-1}^{n+1} + (1 + d_y)\, u_{ij}^{n+1} - \frac{d_y}{2} u_{i,j+1}^{n+1} = u_{i,j}^{n+\frac{1}{2}} + \frac{d_x}{2}\left(u_{i-1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i+1,j}^{n+\frac{1}{2}}\right) \tag{29}$$

Note that equation (28) is implicit in the x-direction and explicit in the y-direction, known as the x-sweep. The solution of (28) provides the data for (29) so that the y-sweep can be carried out in which the solution is implicit in the y-direction and explicit in the x-direction. The ADI formulation can be shown to be an *approximate factorization method* based on the Crank–Nicolson scheme. To show this, let us introduce the following compact notation:

$$\delta_x^2 u_{ij} \equiv u_{i+1,j} - 2u_{ij} + u_{i-1,j} \qquad \text{and} \qquad \delta_y^2 u_{ij} \equiv u_{i,j+1} - 2u_{ij} + u_{i,j-1}$$

The Crank-Nicolson equation (24) can now be written as

$$\frac{u_{ij}^{n+1} - u_{ij}^n}{\Delta t} = \frac{\alpha}{2}\left[\left(\frac{\delta_x^2 u_{ij}^n}{\Delta x^2} + \frac{\delta_y^2 u_{ij}^n}{\Delta y^2}\right) + \left(\frac{\delta_x^2 u_{ij}^{n+1}}{\Delta x^2} + \frac{\delta_y^2 u_{ij}^{n+1}}{\Delta y^2}\right)\right] \tag{30}$$

we may rewrite equation (30) as

$$u_{ij}^{n+1} - u_{ij}^n = \frac{1}{2}\left[\left(d_x \delta_x^2 u_{ij}^n + d_y \delta_y^2 u_{ij}^n\right) + \left(d_x \delta_x^2 u_{ij}^{n+1} + d_y \delta_y^2 u_{ij}^{n+1}\right)\right]$$

or

$$\left[1 - \frac{1}{2}\left(d_x \delta_x^2 + d_y \delta_y^2\right)\right] u_{ij}^{n+1} = \left[1 + \frac{1}{2}\left(d_x \delta_x^2 + d_y \delta_y^2\right)\right] u_{ij}^n \tag{31}$$

To compare equation (31) with the ADI formulation, we use the compact notations to rewrite the ADI equations (26) and (27) as

$$\frac{u_{ij}^{n+\frac{1}{2}} - u_{ij}^n}{\Delta t/2} = \alpha\left(\frac{\delta_x^2 u_{ij}^{n+\frac{1}{2}}}{\Delta x^2} + \frac{\delta_y^2 u_{ij}^n}{\Delta y^2}\right) \tag{32}$$

$$\frac{u_{ij}^{n+1} - u_{ij}^{n+\frac{1}{2}}}{\Delta t/2} = \alpha\left(\frac{\delta_x^2 u_{ij}^{n+\frac{1}{2}}}{\Delta x^2} + \frac{\delta_y^2 u_{ij}^{n+1}}{\Delta y^2}\right) \tag{33}$$

Rearranging equations (32) and (33)

$$\left(1 - \frac{1}{2} d_x \delta_x^2\right) u_{ij}^{n+\frac{1}{2}} = \left(1 + \frac{1}{2} d_y \delta_y^2\right) u_{ij}^n \tag{34}$$

$$\left(1 - \frac{1}{2} d_y \delta_y^2\right) u_{ij}^{n+1} = \left(1 + \frac{1}{2} d_x \delta_x^2\right) u_{ij}^{n+\frac{1}{2}} \tag{35}$$

and eliminating $u_{ij}^{n+\frac{1}{2}}$ between (34) and (35),

$$\left(1-\frac{1}{2}d_x\delta_x^2\right)\left(1-\frac{1}{2}d_y\delta_y^2\right)u_{ij}^{n+1} = \left(1+\frac{1}{2}d_x\delta_x^2\right)\left(1+\frac{1}{2}d_y\delta_y^2\right)u_{ij}^{n} \tag{36}$$

or

$$\left[1-\frac{1}{2}\left(d_x\delta_x^2+d_y\delta_y^2\right)+\frac{1}{4}d_xd_y\delta_x^2\delta_y^2\right]u_{ij}^{n+1} = \left[1+\frac{1}{2}\left(d_x\delta_x^2+d_y\delta_y^2\right)+\frac{1}{4}d_xd_y\delta_x^2\delta_y^2\right]u_{ij}^{n} \tag{37}$$

Compared to (31), equation (37) has the additional term $\frac{1}{4}d_xd_y\delta_x^2\delta_y^2\left(u_{ij}^{n+1}-u_{ij}^{n}\right)$ which represent errors with respect to the original Crank–Nicolson scheme. However, these error terms, proportional to Δt^2, are of the same order as the truncation error (of Crank–Nicolson scheme) and hence do not affect the overall accuracy of the scheme. Therefore, it is seen that the Crank–Nicolson scheme of (31) can be approximated by (37), which in turn can be factored as (36) and then split as (34) and (35). Equation (36) is known as the a *approximate factorization* of (31).

Splitting or fractional step method

In the *fractional step method*, introduced by Yanenko [3], the original multidimensional equation is split into a series of one-dimensional equations and then solve them sequentially TDMA. This formulation can also be considered as a approximate factorization method. The method provides the following discretized equations for two-dimensional diffusion equation:

$$\frac{u_{ij}^{n+\frac{1}{2}}-u_{ij}^{n}}{\Delta t/2} = \frac{\alpha}{\Delta x^2}\left(u_{i+1,j}^{n+\frac{1}{2}}-2u_{ij}^{n+\frac{1}{2}}+u_{i-1,j}^{n+\frac{1}{2}}\right) \tag{38}$$

$$\frac{u_{ij}^{n+1}-u_{ij}^{n+\frac{1}{2}}}{\Delta t/2} = \frac{\alpha}{\Delta y^2}\left(u_{i,j+1}^{n+1}-2u_{ij}^{n+1}+u_{i,j-1}^{n+1}\right) \tag{39}$$

This scheme is of the order of $O\left(\Delta t,\Delta x^2,\Delta y^2\right)$ and is unconditionally stable. The temporal accuracy can be made second-order by using a Crank–Nicolson scheme within each fractional step.

$$\frac{u_{ij}^{n+\frac{1}{2}}-u_{ij}^{n}}{\Delta t/2} = \frac{\alpha}{2\Delta x^2}\left(u_{i+1,j}^{n+\frac{1}{2}}-2u_{ij}^{n+\frac{1}{2}}+u_{i-1,j}^{n+\frac{1}{2}}+u_{i+1,j}^{n}-2u_{ij}^{n}+u_{i-1,j}^{n}\right) \tag{40}$$

$$\frac{u_{ij}^{n+1}-u_{ij}^{n+\frac{1}{2}}}{\Delta t/2} = \frac{\alpha}{2\Delta y^2}\left(u_{i,j+1}^{n+1}-2u_{ij}^{n+1}+u_{i,j-1}^{n+1}+u_{i,j+1}^{n+\frac{1}{2}}-2u_{ij}^{n+\frac{1}{2}}+u_{i,j-1}^{n+\frac{1}{2}}\right) \tag{41}$$

This scheme is of the order of $O\left(\Delta t^2,\Delta x^2,\Delta y^2\right)$ and is unconditionally stable.

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