Streamfunction-Vorticity Formulation

A. Salih
Department of Aerospace Engineering
Indian Institute of Space Science and Technology, Thiruvananthapuram
– March 2013 –

The streamfunction-vorticity formulation was among the first unsteady, incompressible Navier–Stokes algorithms. The original finite difference algorithm was developed by Fromm [1] at Los Alamos laboratory. For incompressible two-dimensional flows with constant fluid properties, the Navier–Stokes equations can be simplified by introducing the streamfunction ψ and vorticity $\overline{\omega}$ as dependent variables. The vorticity vector at a point is defined as twice the angular velocity and is

$$\overline{\omega} = \nabla \times \overline{V} \tag{1}$$

which, for two-dimensional flow in x-y plane, is reduced to

$$\omega_z = \overline{\omega} \cdot \hat{k} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{2}$$

For two-dimensional, incompressible flows, a scalar function may be defined in such a way that the continuity equation is identically satisfied if the velocity components, expressed in terms of such a function, are substituted in the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

Such a function is known as the streamfunction, and is given by

$$\overline{V} = \nabla \times \psi \hat{k} \tag{4}$$

In Cartesian coordinate system, the above relation becomes

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \tag{5}$$

Lines of constant ψ are streamlines (lines which are everywhere parallel to the flow), giving this variable its name.

Now, a Poisson equation for ψ can be obtained by substituting the velocity components, in terms of streamfunction, in the equation (2). Thus, we have

$$\nabla^2 \psi = -\omega \tag{6}$$

where the subscript z is dropped from ω_z. This is a kinematic equation connecting the streamfunction and the vorticity. So if we can find an equation for ω we will have obtained a formulation that automatically produces divergence-free velocity fields.

Finally, by taking the curl of the general Navier–Stokes equation, we obtain the following Helmholtz equation:

$$\frac{\partial \overline{\omega}}{\partial t} + \overline{V} \cdot \nabla \overline{\omega} = \nabla \times \overline{f}_{\mathrm{e}} + \overline{\omega} \cdot \nabla \overline{V} - \overline{\omega} \nabla \cdot \overline{V} + \frac{1}{\rho^2} \nabla \rho \times \nabla p + \nabla \times \left(\frac{1}{\rho} \nabla \cdot \overline{\overline{\tau}} \right) \tag{7}$$

For a Newtonian fluid with constant kinematic viscosity coefficient ν, the viscous stress term reduces to the Laplacian of the vorticity

$$\nabla \times \left(\frac{1}{\rho} \nabla \cdot \overline{\overline{\tau}} \right) = \nu \nabla^2 \overline{\omega}$$

For incompressible flow with constant density, the third and fourth terms on the right-hand side become zero. Further, in the absence of body force, Helmholtz equation reduces to

$$\frac{\partial \overline{\omega}}{\partial t} + \overline{V} \cdot \nabla \overline{\omega} = \overline{\omega} \cdot \nabla \overline{V} + \nu \nabla^2 \overline{\omega} \tag{8}$$

For two-dimensional flows, the term, $\overline{\omega} \cdot \nabla \overline{V} = 0$ by continuity equation (3), and the Helmholtz equation further reduces to a form

$$\frac{\partial \omega}{\partial t} + \overline{V} \cdot \nabla \omega = \nu \nabla^2 \omega \tag{9}$$

This parabolic PDE is called the *vorticity transport equation*.

An alternate approach to derive the vorticity transport equation from the scalar form of momentum is by cross-differentiation. The two-dimensional Navier–Stokes equation for incompressible flow without body force term is given by

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{10}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \tag{11}$$

Differentiating equation (10) and (11) with respect to y and x respectively yield

$$\frac{\partial^2 u}{\partial y \partial t} + \frac{\partial u}{\partial y} \frac{\partial u}{\partial x} + u \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} + v \frac{\partial^2 u}{\partial y^2} = -\frac{1}{\rho} \frac{\partial^2 p}{\partial y \partial x} + \nu \left(\frac{\partial^3 u}{\partial y \partial x^2} + \frac{\partial^3 u}{\partial y^3} \right) \tag{12}$$

$$\frac{\partial^2 v}{\partial x \partial t} + \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + u \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} + v \frac{\partial^2 v}{\partial x \partial y} = -\frac{1}{\rho} \frac{\partial^2 p}{\partial x \partial y} + \nu \left(\frac{\partial^3 v}{\partial x^3} + \frac{\partial^3 v}{\partial x \partial y^2} \right) \tag{13}$$

Subtract (12) from (13) to obtain (assume sufficient smoothness to permit changing the order of differentiation)

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + u \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + v \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \\ = \nu \left[\frac{\partial^2}{\partial x^2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \right] \end{aligned}$$

Note that the fourth term on the left-hand side is zero by continuity. The substitution of vorticity defined by (2), we obtain the vorticity transport equation

$$\frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} = \nu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) \tag{14}$$

The relation connecting the streamfunction and vorticity (6) is listed below:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega \tag{15}$$

Equations (14) and (15) form the system PDEs for streamfunction-vorticity formulation. The pressure does not appear in either of these equations i.e. it has been eliminated as a dependent variable. Thus the Navier–Stokes equations have been replaced by a set of just two partial differential equations, in place of the three for the velocity components and pressure. It is instructive to note that the absence of an explicit evolution equation for pressure in the system of equation is reflected in the absence of an evolution equation for streamfunction. Further, the mixed elliptic-parabolic nature of the original Navier–Stokes system is clearly seen from the new system of equation.

The two equations are coupled through the appearance of u and v (which are derivatives of ψ) in the vorticity equation and by the vorticity ω acting as the source term in the Poisson equation for ψ. The velocity components are obtained by differentiating the streamfunction. If the pressure field is desired it can be obtained a posteriori by solving the Pressure Poisson Equation (PPE).

Pressure Poisson Equation

The PPE can be derived by taking the divergence of vector form of momentum equation

$$\frac{\partial \overline{V}}{\partial t} + \left(\overline{V} \cdot \nabla\right) \overline{V} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \overline{V}$$

or substituting the component form of the momentum equations (10) and (11) into the scalar form of continuity equation.

Differentiating (10) with respect to x to provide

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial x \partial y} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x^2} + \nu \frac{\partial}{\partial x}\left(\nabla^2 u\right) \tag{16}$$

Similarly, differentiating (11) with respect to y to provide

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial t}\right) + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial y \partial x} + \left(\frac{\partial v}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial y^2} + \nu \frac{\partial}{\partial y}\left(\nabla^2 v\right) \tag{17}$$

Addition of equations (16) and (17) yields

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + 2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right) + v\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}\right) \\ = -\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) + \nu\left[\frac{\partial}{\partial x}\left(\nabla^2 u\right) + \frac{\partial}{\partial y}\left(\nabla^2 v\right)\right] \end{aligned} \tag{18}$$

Note that first, fifth, and sixth terms each contain the continuity equation and therefore disappear:

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \\ &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \\ &\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \end{aligned}$$

The right-hand is now rearranged to provide

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial}{\partial y}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$

Thus, the equation (18) reduces to

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}=-\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right) \tag{19}$$

Now, the left-hand side can be further reduced as follows

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2-2\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}=2\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right)$$

Therefore the PPE can be written as

$$\nabla^2 p=2\rho\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right) \tag{20}$$

The PPE can also be written in terns of streamfunction using the relations in (5)

$$\nabla^2 p=2\rho\left[\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2}-\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2\right] \tag{21}$$

Poisson equation for pressure is an elliptic equation, showing the elliptic nature of pressure in incompressible flows. For a steady flow problem, the PPE is solved only once, i.e., after the steady state values of ω and ψ have been computed. To solve PPE, boundary conditions for pressure are required. On a solid boundary, boundary values of pressure obtained by tangential momentum equation to the fluid adjacent to the wall surface. For a wall located at $y=0$ in Cartesian coordinate system, the tangential momentum equation (x-momentum equation) reduces to

$$\left.\frac{\partial p}{\partial x}\right)_{\text{wall}}=\mu\left.\frac{\partial^2 u}{\partial y^2}\right)_{\text{wall}}=-\mu\left.\frac{\partial\omega}{\partial y}\right)_{\text{wall}} \tag{22}$$

since $\partial v/\partial x=0$ along such a wall. Equation (22) can be discretized as

$$\frac{p_{i+1,1}-p_{i-1,1}}{2\Delta x}=-\mu\frac{-3\omega_{i,1}+4\omega_{i,2}-\omega_{i,3}}{2\Delta y} \tag{23}$$

where we have used second-order accurate one-sided forward difference formula to approximate the derivative $\partial\omega/\partial y$. Note that, in order to apply equation (23), the pressure must be known for at least one point on the wall surface. Equation (23) when applied to the grid point (2,1) becomes

$$\frac{p_{3,1}-p_{1,1}}{2\Delta x}=-\mu\frac{-3\omega_{2,1}+4\omega_{2,2}-\omega_{2,3}}{2\Delta y}$$

The pressure $p_{2,1}$ at the adjacent point can be determined using a first-order expression for $\partial p/\partial x$ as given below:

$$\frac{p_{2,1}-p_{1,1}}{\Delta x}=-\mu\frac{-3\omega_{2,1}+4\omega_{2,2}-\omega_{2,3}}{2\Delta y}$$

Thereafter, equation (23) can be used to find the pressure at all other bottom wall points.

Discretization of governing equations

The governing equations for incompressible, two-dimensional Navier–Stokes equation using streamfunction-vorticity are derived in the previous Section. Essentially, the system is composed of the vorticity transport equation (9) and the Poisson equation for streamfunction (15).

As mentioned earlier, the vorticity transport equation is a parabolic PDE and thus any suitable method for parabolic PDE can be used to solve equation (14). Here we use the explicit FTCS scheme, where Euler forward difference for temporal derivative and central differences for space derivatives are used.

$$\frac{\omega_{ij}^{n+1}-\omega_{ij}^{n}}{\Delta t}+u_{ij}\left(\frac{\omega_{i+1,j}^{n}-\omega_{i-1,j}^{n}}{2\Delta x}\right)+v_{ij}\left(\frac{\omega_{i,j+1}^{n}-\omega_{i,j-1}^{n}}{2\Delta y}\right)$$
$$=\nu\left(\frac{\omega_{i+1,j}^{n}-2\omega_{ij}^{n}+\omega_{i-1,j}^{n}}{\Delta x^2}+\frac{\omega_{i,j+1}^{n}-2\omega_{ij}^{n}+\omega_{i,j-1}^{n}}{\Delta y^2}\right) \tag{24}$$

The stability conditions for the FTCS scheme are

$$d\equiv\frac{\nu\Delta t}{\Delta x^2}+\frac{\nu\Delta t}{\Delta y^2}\le\frac{1}{2}\qquad\text{or}\qquad Re_{\Delta x}c_x+Re_{\Delta y}c_y\le 2 \tag{25}$$

where

$$Re_{\Delta x}=\frac{u\Delta x}{\nu}\qquad Re_{\Delta y}=\frac{v\Delta y}{\nu}\qquad c_x=\frac{u\Delta t}{\Delta x}\qquad c_y=\frac{v\Delta t}{\Delta y}$$

Recall that CDS approximation of convective terms does not model the physics of the problem accurately in that it does not correctly represent the directional influence of a disturbance. Therefore, the use of upwind type differencing scheme may be more appropriate in particular if the flow field is convection dominated. However, first-order upwind scheme is too diffusive and may not be suitable for practical applications. Thus, we use the second-order upwind scheme for the discretization of convection terms. The discretized vorticity transport equation can be written as

$$\frac{\omega_{ij}^{n+1}-\omega_{ij}^{n}}{\Delta t}+u_{ij}\left(\frac{\omega_{i+1,j}^{n}-\omega_{i-1,j}^{n}}{2\Delta x}\right)+q\left(u^{+}\omega_x^{-}+u^{-}\omega_x^{+}\right)+v_{ij}\left(\frac{\omega_{i,j+1}^{n}-\omega_{i,j-1}^{n}}{2\Delta y}\right)$$
$$+q\left(v^{+}\omega_y^{-}+v^{-}\omega_y^{+}\right)=\nu\left(\frac{\omega_{i+1,j}^{n}-2\omega_{ij}^{n}+\omega_{i-1,j}^{n}}{\Delta x^2}+\frac{\omega_{i,j+1}^{n}-2\omega_{ij}^{n}+\omega_{i,j-1}^{n}}{\Delta y^2}\right) \tag{26}$$

where

$$u^{-}\equiv\min(u_{ij}^{n},0)\qquad u^{+}\equiv\max(u_{ij}^{n},0)$$
$$v^{-}\equiv\min(v_{ij}^{n},0)\qquad v^{+}\equiv\max(v_{ij}^{n},0)$$

$$\omega_x^{-}\equiv\frac{\omega_{i-2,j}^{n}-3\omega_{i-1,j}^{n}+3\omega_{ij}^{n}-\omega_{i+1,j}^{n}}{3\Delta x}\qquad \omega_x^{+}\equiv\frac{\omega_{i-1,j}^{n}-3\omega_{ij}^{n}+3\omega_{i+1,j}^{n}-\omega_{i+2,j}^{n}}{3\Delta x}$$
$$\omega_y^{-}\equiv\frac{\omega_{i,j-2}^{n}-3\omega_{i,j-2}^{n}+3\omega_{ij}^{n}-\omega_{i,j+1}^{n}}{3\Delta x}\qquad \omega_y^{+}\equiv\frac{\omega_{i,j-1}^{n}-3\omega_{ij}^{n}+3\omega_{i,j+1}^{n}-\omega_{i,j+2}^{n}}{3\Delta x}$$

It may be noted that $q=0.5$ represents the third-order accurate upwind formula and for other values of q, the modified formula is only second-order accurate. Also, $q=0$ corresponds to the central difference scheme.

It may also be noted that, when the discretized equation is applied to grid points adjacent to boundaries, grid points $(i-2, j)$, $(i+2, j)$, $(i, j-2)$ etc. lie outside the domain and we have no information about the value of ω on such points. This problem can be avoided by setting $q = 0$ for grid points immediately adjacent to the boundary.

The streamfunction equation (15) is solved at every time step using an appropriate numerical scheme. Since it is an elliptic equations, we use the standard central differencing scheme for discretization of second order spatial derivatives. The discretized equation is given as follows

$$\frac{\psi_{i+1,j}^{n+1} - 2\psi_{ij}^{n+1} + \psi_{i-1,j}^{n+1}}{\Delta x^2} + \frac{\psi_{i,j+1}^{n+1} - 2\psi_{ij}^{n+1} + \psi_{i,j-1}^{n+1}}{\Delta y^2} = -\omega_{ij}^{n+1} \tag{27}$$

Boundary conditions

The solution of vorticity transport equation and stream function equation requires that appropriate vorticity and streamfunction boundary conditions are specified at the boundaries. The specification of these boundary conditions is extremely important since it directly affects the stability and accuracy of the solution. Since the flow is parallel to a solid boundary, solid boundaries and symmetry planes are surfaces of constant streamfunction. However, neither vorticity nor its derivatives at the boundary are usually known in advance. Therefore a set of boundary conditions must be constructed. We will consider this in the context of a classical problem which has wall boundaries surrounding the entire computational region, the so-called 'lid-driven cavity' problem depicted in figure 1. We consider the domain included in a square of unit length, with $0 \le x, y \le 1$, where the upper boundary (the lid) at $y = 1$, moves with a constant velocity $U = 1$. The Reynolds number based on the size of the domain, the velocity of the moving wall, density $\rho = 1$ and viscosity $\mu = 0.005$ is $Re = 200$. The boundary

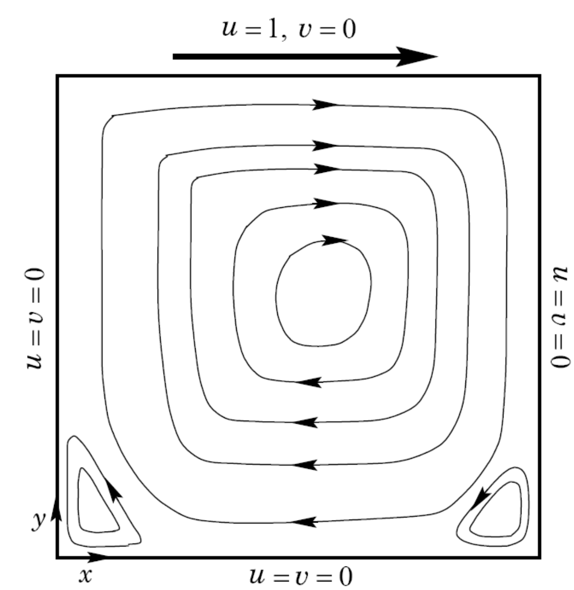

Figure 1: Lid-driven cavity flow - typical streamline pattern.

conditions are set as follows

$$\begin{aligned} u(x,0) &= 0 \qquad & v(x,0) &= 0 \\ u(x,1) &= 1 \qquad & v(x,1) &= 0 \\ u(0,y) &= 0 \qquad & v(0,y) &= 0 \\ u(1,y) &= 0 \qquad & v(1,y) &= 0 \end{aligned}$$

Since flow is parallel to the walls of the cavity, walls may be treated as streamline. Thus, the streamfunction value on the wall streamline is set as a constant. That is $\psi = c$, where c is an arbitrary constant, which may be set equal to zero.

Let us examine the application of boundary conditions on a solid wall. Since streamfunction is a constant along a wall, all the derivatives of streamfunction along the wall vanish. Hence, the Poisson equation for streamfunction (15) reduces to

$$\left.\frac{\partial^2 \psi}{\partial n^2}\right|_{\text{wall}} = -\omega_{\text{wall}} \tag{28}$$

where n is the normal direction. For a wall located at $x = 0$ (left wall), equation (28) takes the form

$$\omega_{1,j} = -\left.\frac{\partial^2 \psi}{\partial x^2}\right|_{1,j} \tag{29}$$

To obtain a finite difference approximation for the second-order derivative in the equation above, consider the Taylor series expansion

$$\psi_{2,j} = \psi_{1,j} + \left.\frac{\partial \psi}{\partial x}\right|_{1,j} \Delta x + \left.\frac{\partial^2 \psi}{\partial x^2}\right|_{1,j} \frac{\Delta x^2}{2} + \cdots$$

Along left wall

$$\left.\frac{\partial \psi}{\partial x}\right|_{1,j} = -v_{1,j}$$

Therefore, Taylor series expansion can be rearranged to get

$$\left.\frac{\partial^2 \psi}{\partial x^2}\right|_{1,j} = \frac{2\left(\psi_{2,j} - \psi_{1,j}\right)}{\Delta x^2} + \frac{2v_{1,j}}{\Delta x} \tag{30}$$

Substitution of equation (30) into (29) yields

$$\omega_{1,j} = \frac{2\left(\psi_{1,j} - \psi_{2,j}\right)}{\Delta x^2} - \frac{2v_{1,j}}{\Delta x} \tag{31}$$

A similar procedure is used to derive the boundary conditions at right, bottom, and top wall. The appropriate expressions are

$$\omega_{M,j} = -\left.\frac{\partial^2 \psi}{\partial x^2}\right|_{M,j} = \frac{2\left(\psi_{M,j} - \psi_{M-1,j}\right)}{\Delta x^2} + \frac{2v_{M,j}}{\Delta x} \tag{32}$$

$$\omega_{i,1} = -\left.\frac{\partial^2 \psi}{\partial y^2}\right|_{i,1} = \frac{2\left(\psi_{i,1} - \psi_{i,2}\right)}{\Delta y^2} + \frac{2u_{i,1}}{\Delta y} \tag{33}$$

$$\omega_{i,N} = -\left.\frac{\partial^2 \psi}{\partial y^2}\right|_{i,N} = \frac{2\left(\psi_{i,N} - \psi_{i,N-1}\right)}{\Delta y^2} - \frac{2u_{i,N}}{\Delta y} \tag{34}$$

It is also possible to approximate the second-order derivatives with second order expressions. These are given by

$$\omega_{1,j} = \frac{7\psi_{1,j} - 8\psi_{2,j} + \psi_{3,j}}{2\Delta x^2} - \frac{3v_{1,j}}{\Delta x} \quad (35)$$

$$\omega_{M,j} = \frac{-7\psi_{M,j} + 8\psi_{M-1,j} - \psi_{M-2,j}}{2\Delta x^2} + \frac{3v_{M,j}}{\Delta x} \quad (36)$$

$$\omega_{i,1} = \frac{7\psi_{i,1} - 8\psi_{i,2} + \psi_{i,3}}{2\Delta y^2} + \frac{3u_{i,1}}{\Delta y} \quad (37)$$

$$\omega_{i,N} = \frac{-7\psi_{i,N} + 8\psi_{i,N-1} - \psi_{i,N-2}}{2\Delta y^2} - \frac{3u_{i,N}}{\Delta y} \quad (38)$$

It may be noted that the first-order expressions for ω at the boundaries often gives better results than higher-order expressions which are susceptible to instabilities at higher Reynolds numbers.

Other type of boundaries

The specification of appropriate values for ψ and ω at other type of boundaries such as far-field, symmetry lines, inflow, and outflow planes is extremely important and care must be taken to ensure that the physics of the problem is correctly modeled.

Far-field

For a true far-field boundary which is set parallel to the freestream, the boundary represents a streamline. Therefore a constant value for the streamfunction along this boundary can be specified. However, the assignment of a value for the streamfunction along various boundaries must be consistent with respect to the continuity equation. Recall that the difference between the values of streamfunction represent volumetric flow.

Line of symmetry

When the flow is truly symmetrical, the axis of symmetry can be considered a streamline. therefore the value of streamfunction along this boundary can be specified. Obviously, the velocity component normal to the the symmetry boundary would be zero, whereas the streamwise component is extrapolated from the interior solution.

Inflow boundary

At the inflow boundary, the values of streamfunction are determined by the following method. Let us assume that the inflow boundary is on the left-side (west). On the part below and above the inflow section, we can respectively assign $\psi = c_1$ and $\psi = c_2$. For convenience we may set $c_1 = 0$ and $c_2 = c$, that is

$$\psi_L = 0 \qquad \psi_U = c$$

Since $\psi_L - \psi_U$ represents the volume flow rate between the streamlines corresponding to ψ_L and ψ_U, the constant$c$ can be so selected that it is equal to the volume flow rate through inflow boundary per unit depth normal to the paper.

In general, the velocity vector may be inclined to the inflow boundary. Therefore, we may write

$$\left.\frac{\partial \psi}{\partial x}\right|_{1,j} = -v_{1,j}$$

The first-order derivative on the left-hand side can be approximated by the one-sided forward difference formula to obtain,

$$\frac{-3\psi_{1,j} + 4\psi_{2,j} - \psi_{3,j}}{2\Delta x} = -v_{1,j}$$

Therefore,

$$\psi_{1,j} = \frac{4\psi_{2,j} - \psi_{3,j} + 2\Delta x v_{1,j}}{3} \tag{39}$$

The vorticity at the inflow boundary may be determined in the following way:

$$\begin{aligned}\omega &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \\ &= -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial u}{\partial y}\end{aligned}$$

On the inflow section

$$\omega_{1,j} = -\left.\frac{\partial^2 \psi}{\partial x^2}\right|_{1,j} - \left.\frac{\partial u}{\partial y}\right|_{1,j}$$

which may be approximated (using formula (31)) as

$$\omega_{1,j} = \frac{2\left(\psi_{1,j} - \psi_{2,j}\right)}{\Delta x^2} - \frac{2v_{1,j}}{\Delta x} - \frac{u_{1,j+1} - u_{1,j-1}}{2\Delta y} \tag{40}$$

This is first-order approximation, and a second-order approximation is given by

$$\omega_{1,j} = \frac{7\psi_{1,j} - 8\psi_{2,j} + \psi_{3,j}}{2\Delta x^2} - \frac{3v_{1,j}}{\Delta x} - \frac{u_{1,j+1} - u_{1,j-1}}{2\Delta y} \tag{41}$$

Outflow boundary

The method we discussed for the inflow boundary can be used for outflow boundary as well. Let us assume that the inflow boundary is on the right-side (east).

In general, the velocity vector may be inclined to the outflow boundary. Therefore, we may write

$$\left.\frac{\partial \psi}{\partial x}\right|_{M,j} = -v_{M,j}$$

The first-order derivative on the left-hand side can be approximated by the one-sided backward difference formula to obtain,

$$\frac{3\psi_{M,j} - 4\psi_{M-1,j} + \psi_{M-2,j}}{2\Delta x} = -v_{M,j}$$

Therefore,

$$\psi_{M,j} = \frac{4\psi_{M-1,j} - \psi_{M-2,j} - 2\Delta x v_{M,j}}{3} \tag{42}$$

On the inflow section

$$\omega_{M,j} = -\left.\frac{\partial^2 \psi}{\partial x^2}\right|_{M,j} - \left.\frac{\partial u}{\partial y}\right|_{M,j}$$

which may be approximated (using formula (32)) as

$$\omega_{1,j} = \frac{2\left(\psi_{M,j} - \psi_{M-1,j}\right)}{\Delta x^2} + \frac{2v_{M,j}}{\Delta x} - \frac{u_{M,j+1} - u_{M,j-1}}{2\Delta y} \tag{43}$$

This is first-order approximation, and a second-order approximation is given by

$$\omega_{1,j} = \frac{-7\psi_{M,j} + 8\psi_{M-1,j} - \psi_{M-2,j}}{2\Delta x^2} + \frac{3v_{M,j}}{\Delta x} - \frac{u_{M,j+1} - u_{M,j-1}}{2\Delta y} \tag{44}$$

Algorithm for streamfunction-vorticity formulation

A solution algorithm for computing evolution of incompressible, two-dimensional flow using streamfunction-vorticity formulation is given as follows:

1. Initialize the velocity field and compute the associated vorticity field and streamfunction field using equations (2) and (15).
2. Compute the boundary conditions for vorticity.
3. Solve the vorticity transport equation (14) to compute the vorticity at new time step; any standard time marching scheme may be used for this purpose.
4. Solve the Poisson equation for streamfunction (15) to compute the streamfunction field at new time step; any iterative scheme for elliptic equations may be used.
5. Compute the velocity field at new time step using the relations (5).
6. Return to step 2 and repeat the computation for another time step.

The vorticity-streamfunction approach has seen considerable use for two-dimensional incompressible flows. It has become less popular in recent years because its extension to three-dimensional flows is difficult. Both the vorticity and streamfunction become three-component vectors in three dimensions so one has a system of six partial differential equations in place of the four that are necessary in a velocity-pressure formulation. It also inherits the difficulties in dealing with variable fluid properties, compressibility, and boundary conditions that were described above for two dimensional flows.

References

1. Fromm, J. E., The Time Dependent Flow of an Incompressible Viscous Fluid, *Meth. Comput. Phys.*, **3**, 345-382 (1964).
2. Hoffmann, K. A. and Chiang, S. T., *Computational Fluid Dynamics for Engineers*, Vol. I, 4th ed., Engineering Education Systems (2000).
3. Pletcher, R. H., Tannehill, J. C., and Anderson, D. A., *Computational Fluid Dynamics and Heat Transfer*, 3rd ed., Taylor & Francis (2011).
4. Roache, P. J., *Fundamentals of Computational Fluid Dynamics*, 2nd ed., Hermosa Pub. (1998).