DEFINITION Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

be an $m \times n$ matrix. The rows of A,

$$\begin{aligned} \mathbf{v}_1 &= (a_{11}, a_{12}, \ldots, a_{1n}) \\ \mathbf{v}_2 &= (a_{21}, a_{22}, \ldots, a_{2n}) \\ &\vdots \\ \mathbf{v}_m &= (a_{m1}, a_{m2}, \ldots, a_{mn}), \end{aligned}$$

considered as vectors in R^n, span a subspace of R^n, called the **row space** of A. Similarly, the columns of A,

$$\mathbf{w}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \qquad \mathbf{w}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \qquad \mathbf{w}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

considered as vectors in R^m, span a subspace of R^m, called the **column space** of A.

THEOREM 6.10 *If A and B are two $m \times n$ row equivalent matrices, then the row spaces of A and B are equal.*

Proof If A and B are row equivalent, then the rows of B are obtained from those of A by a finite number of the three elementary row operations. Thus each row of B is a linear combination of the rows of A. Hence the row space of B is contained in the row space of A. Similarly, A can be obtained from B by a finite number of the three elementary row operations, so the row space of A is contained in the row space of B. Hence the row spaces of A and B are equal. ■

Remark For a related result see Theorem 1.7 in Section 1.6, where we showed that if two augmented matrices are row equivalent, then their corresponding linear systems have the same solutions.

It follows from Theorem 6.10 that if we take a given matrix A and find its reduced row echelon form B, then the row spaces of A and B are equal. Furthermore, recall from Exercise T.8 in Section 6.3 that the nonzero rows of a matrix that is in reduced row echelon form are linearly independent and thus form a basis for its row space. We can use this method to find a basis for a vector space spanned by a given set of vectors in R^n, as illustrated in Example 1.

EXAMPLE 1 Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where

$$\mathbf{v}_1 = (1, -2, 0, 3, -4), \qquad \mathbf{v}_2 = (3, 2, 8, 1, 4),$$
$$\mathbf{v}_3 = (2, 3, 7, 2, 3), \qquad \mathbf{v}_4 = (-1, 2, 0, 4, -3),$$

and let V be the subspace of R^5 given by $V = \operatorname{span} S$. Find a basis for V.

Solution Note that V is the row space of the matrix A whose rows are the given vectors

$$A = \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix}.$$

Using elementary row operations, we find that A is row equivalent to the matrix (verify)

$$B = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which is in reduced row echelon form. The row spaces of A and B are identical, and a basis for the row space of B consists of the nonzero rows of B. Hence

$$\mathbf{w}_1 = (1, 0, 2, 0, 1), \quad \mathbf{w}_2 = (0, 1, 1, 0, 1), \quad \text{and} \quad \mathbf{w}_3 = (0, 0, 0, 1, -1)$$

form a basis for V. ■

Remark Observe in Example 1 that the rows of the matrices A and B are different, but their row spaces are the same.

We may now summarize the method used in Example 1 to find a basis for the subspace V of R^n given by $V = \text{span } S$, where S is a set of vectors in R^n.

> The procedure for finding a basis for the subspace V of R^n given by $V = \text{span } S$, where $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in R^n that are given in row form, is as follows.
>
> ***Step 1.*** Form the matrix
>
> $$A = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_k \end{bmatrix}$$
>
> whose rows are the given vectors in S.
>
> ***Step 2.*** Transform A to reduced row echelon form, obtaining the matrix B.
>
> ***Step 3.*** The nonzero rows of B form a basis for V.