

Description of Stress at a Point

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 – March 2012 –

The stress at a point in a material needs nine components for its complete description because *two* directions (and therefore, *two* free indices) are involved in its description. One of the directions specifies the orientation of the *surface* on which the stress is acting, and the other specifies the direction of the *force* on that surface. Let us look at it in more detail.

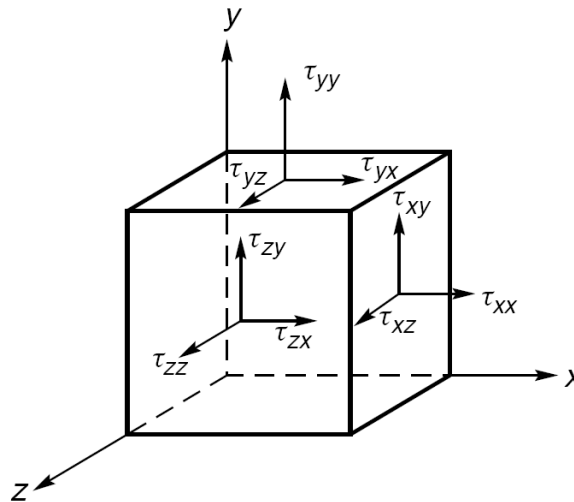


Figure 1: Stress components on an infinitesimal rectangular parallelepiped.

Consider an infinitesimal rectangular parallelepiped with faces perpendicular to the coordinate axes as shown in figure 1. On each face there is a normal stress and a tangential stress. The tangential stress can be further resolved into two components in the directions of the axes. The complete stress matrix is given below:

$$\bar{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

The sign convention for the stress is that, on a surface whose outward normal points in the positive direction of a coordinate axis, the normal and shear stresses are positive, if they point in the positive direction of the axes. The $(ij)^{\text{th}}$ component of the stress tensor $\bar{\tau}$ is given by τ_{ij} . The first index of τ_{ij} indicate the direction of the normal to the surface on which the stress is considered, and the second index indicates the direction in which the stress acts. The diagonal elements τ_{xx} , τ_{yy} , τ_{zz} are the normal stresses and the off-diagonal elements are the tangential or shear stresses. Although a parallelepiped is shown, its size actually is zero and the figure really shows the stresses on three orthogonal planes passing through a point.

Stress tensor is symmetric

An important property of $\bar{\bar{\tau}}$ is its symmetry, i.e. $\tau_{ji} = \tau_{ij}$. this can be proved by the application of *angular momentum principle* to a differential fluid element, taken as a rectangular parallelepiped. The origin of the coordinate system is assumed to be at the center of the element.

Euler's equation (for a rotating rigid body about z -axis) states that

$$I_z \frac{d\omega_z}{dt} = M_z \quad (1)$$

where

$$I_z = \rho \delta \mathcal{V} \frac{\delta x^2 + \delta y^2}{12} = \frac{\rho}{12} \delta x \delta y \delta z (\delta x^2 + \delta y^2)$$

is the moment of inertia of the fluid element, ω_z is the angular velocity about the positive z -axis, and M_z is the moment of all forces about z -axis.

The moment about the z -axis due to τ_{xy} and τ_{yx} ,

$$(\tau_{xy} \delta y \delta z) \frac{\delta x}{2} - (\tau_{yx} \delta x \delta z) \frac{\delta y}{2} = \frac{1}{2} (\tau_{xy} - \tau_{yx}) \delta x \delta y \delta z$$

The shear stresses acting on the negative x - and y -surfaces contribute as well, so that the total moment about the z -axis is

$$(\tau_{xy} - \tau_{yx}) \delta x \delta y \delta z$$

The substitution of the above quantities into equation (1) gives

$$\frac{\rho}{12} (\delta x^2 + \delta y^2) \frac{d\omega_z}{dt} = \tau_{xy} - \tau_{yx}$$

In order that $\frac{d\omega_z}{dt}$ remains finite, we need $(\tau_{xy} - \tau_{yx}) \rightarrow 0$ when $\delta x, \delta y \rightarrow 0$. It follows that $\tau_{xy} = \tau_{yx}$. Similar considerations establish that

$$\tau_{xy} = \tau_{yx} \quad \tau_{yz} = \tau_{zy} \quad \tau_{zx} = \tau_{xz} \quad (2)$$

The above condition means that the stress tensor must be symmetric:

$$\bar{\bar{\tau}}^T = \bar{\bar{\tau}} \quad \text{or} \quad \tau_{ji} = \tau_{ij}$$

Note that the condition of stress symmetry is not valid if there is a significant body couple per unit mass \bar{c} in the field. In this case, we can easily show that,

$$\bar{\bar{\bar{\epsilon}}} : \bar{\bar{\tau}} - \rho \bar{c} = 0$$

where $\bar{\bar{\bar{\epsilon}}}$ is the third order alternating tensor. This gives a relationship between \bar{c} and the off-diagonal components of $\bar{\bar{\tau}}$, but the stress is clearly not symmetric.

Stress vector on a prescribed surface

On a prescribed surface (unit normal known), the stress is considered as a vector because it can be completely described by just three components. The stress vectors in the planes normal to x , y , and z axes are respectively given as

$$\begin{aligned}\bar{S}^{(x)} &= \tau_{xx}\hat{i} + \tau_{xy}\hat{j} + \tau_{xz}\hat{k} \\ \bar{S}^{(y)} &= \tau_{yx}\hat{i} + \tau_{yy}\hat{j} + \tau_{yz}\hat{k} \\ \bar{S}^{(z)} &= \tau_{zx}\hat{i} + \tau_{zy}\hat{j} + \tau_{zz}\hat{k}\end{aligned}\quad (3)$$

The above relation may be written in a compact form as

$$\begin{bmatrix} \bar{S}^{(x)} \\ \bar{S}^{(y)} \\ \bar{S}^{(z)} \end{bmatrix} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

By the *principle of local stress equilibrium*¹, the stress vector on an arbitrary surface, whose normal \hat{n} , is given by

$$\bar{S}^{(n)} = \bar{S}^{(x)} n_x + \bar{S}^{(y)} n_y + \bar{S}^{(z)} n_z \quad (4)$$

where the unit normal

$$\hat{n} = n_x \hat{i} + n_y \hat{j} + n_z \hat{k}$$

with n_x, n_y, n_z being the direction cosines of the normal \hat{n} .

Now, the stress vector on a surface having unit normal \hat{n} can be resolved into three mutually perpendicular components as

$$\bar{S}^{(n)} = S_x^{(n)} \hat{i} + S_y^{(n)} \hat{j} + S_z^{(n)} \hat{k} \quad (5)$$

From equations (3), (4), and (5) it can be deduced that

$$\begin{aligned}S_x^{(n)} &= \tau_{xx} n_x + \tau_{yx} n_y + \tau_{zx} n_z \\ S_y^{(n)} &= \tau_{xy} n_x + \tau_{yy} n_y + \tau_{zy} n_z \\ S_z^{(n)} &= \tau_{xz} n_x + \tau_{yz} n_y + \tau_{zz} n_z\end{aligned}\quad (6)$$

The above relation may be written in compact form as

$$\begin{bmatrix} S_x^{(n)} \\ S_y^{(n)} \\ S_z^{(n)} \end{bmatrix} = \begin{pmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{pmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

The i^{th} component of the stress vector $\bar{S}^{(n)}$ is given by

$$S_i^{(n)} = \tau_{ji} n_j = \tau_{ij} n_j \quad (\text{since stress tensor is symmetric})$$

¹It states that the surface forces must be in local equilibrium for any arbitrary small volume element. This is true independent of the source or detailed form of the surface forces.

or using vector notation

$$\bar{S}^{(n)} = \bar{\tau}^T \cdot \hat{n} = \bar{\tau} \cdot \hat{n} \quad (7)$$

The above equation states that the dot (or inner) product of the stress tensor $\bar{\tau}$ and the unit outward normal gives the force per unit area (stress) on a surface. Equation (7) is analogous to $u_n = \bar{u} \cdot \hat{n}$, where u_n is the component of the vector \bar{u} along unit normal \hat{n} ; however, whereas u_n is a scalar, $\bar{S}^{(n)}$ in the equation (7) is a vector.

From the above discussion, it follows that the surface stress vector on any arbitrarily oriented surface through a point \bar{x} is completely determined by specification of the nine independent components of the stress tensor $\bar{\tau}$ at that point. The total surface force acting on the entire surface is given by

$$\bar{F}_s = \int_A \bar{S}^{(n)} dA = \int_A \bar{\tau} \cdot \hat{n} dA.$$