Tridiagonal Matrix Algorithm

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A system of simultaneous algebraic equations with nonzero coefficients only on the main diagonal, the lower diagonal, and the upper diagonal is called a *tridiagonal system of equations*. Consider a tridiagonal system of N equations with N unknowns, $u_1, u_2, u_3, \cdots u_N$ as given below:

$$\begin{bmatrix} b_1 & c_1 & & & & & \\ a_2 & b_2 & c_2 & & & & \\ & a_3 & b_3 & c_3 & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & a_{N-1} & b_{N-1} & c_{N-1} \\ & & & & & a_N & b_N \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{N-1} \\ d_N \end{bmatrix} \tag{1}$$

A standard method for solving a system of linear, algebraic equations is gaussian elimination. *Thomas' algorithm*, also called *TriDiagonal Matrix Algorithm* (TDMA) is essentially the result of applying gaussian elimination to the tridiagonal system of equations.

The i^{th} equation in the system may be written as

$$a_i u_{i-1} + b_i u_i + c_i u_{i+1} = d_i \tag{2}$$

where $a_1 = 0$ and $c_N = 0$. Looking at the system of equations, we see that i^{th} unknown can be expressed as a function of $(i+1)^{th}$ unknown. That is

$$u_i = P_i\, u_{i+1} + Q_i \tag{3}$$

$$u_{i-1} = P_{i-1}\, u_i + Q_{i-1} \tag{4}$$

where P_i and Q_i are constants. Note that if all the equations in the system are expressed in this fashion, the coefficient matrix of the system would transform to a an upper triangular matrix.

To determine the constants P_i and Q_i, we plug equation (4) in (2) to yield

$$a_i P_{i-1} u_i + a_i Q_{i-1} + b_i u_i + c_i u_{i+1} = d_i$$

or

$$(b_i + a_i P_{i-1}) u_i + c_i u_{i+1} = d_i - a_i Q_{i-1}$$

or

$$u_i = \frac{-c_i}{b_i + a_i P_{i-1}} u_{i+1} + \frac{d_i - a_i Q_{i-1}}{b_i + a_i P_{i-1}} \tag{5}$$

Comparing equations (3) and (5), we obtain

$$P_i = \frac{-c_i}{b_i + a_i P_{i-1}} \qquad Q_i = \frac{d_i - a_i Q_{i-1}}{b_i + a_i P_{i-1}} \tag{6}$$

These are the recurring relations for the constants P and Q. It shows that P_i can be calculated if P_{i-1} is known. To start the computation, we use the fact that $a_1 = 0$. Now, P_1 and Q_1 can be easily calculated because terms involving P_0 and Q_0 vanish. Therefore,

$$P_1 = \frac{-c_1}{b_1} \qquad Q_1 = \frac{d_1}{b_1} \tag{7}$$

Once the values of P_1 and Q_1 are known, we can use the recurring expressions for P_i and Q_i for all values of i.

Now, to start the back substitution, we use the fact that $c_N = 0$. As a consequence, from equation (6), we have $P_N = 0$, which results in $u_N = Q_N$. Once the value of u_N is known we use equation (3) to obtain $u_{N-1}, u_{N-2}, \cdots u_1$.

A Fortran implementation

The following Fortran code will solve a general tridiagonal system. Note that n is the number of unknowns.

```
      program TDMA
      implicit doubleprecision(a-h,o-z)
      parameter (nd = 100)
      doubleprecision A(nd), B(nd), C(nd), D(nd), X(nd), P(0:nd), Q(0:nd)
c
      A(1) = 0
      C(n) = 0
c
c     forward elimination
      do i = 1, n
         denom = B(i) + A(i)*P(i-1)
         P(i) = -C(i) /denom
         Q(i) = (D(i) - A(i)*Q(i-1)) /denom
      enddo
c
c     back substitution
      do i = n, 1, -1
         X(i) = P(i)*X(i+1) + Q(i)
      enddo
      stop
      end
```