An overview of Cartesian Tensors

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A brief overview of vectors and tensors is given here. A three dimensional vector $\overline{V}$ in Cartesian coordinate system can be written as

$$\overline{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

where u, v, and w are the components of the vector along the three mutually perpendicular directions x, y, and z respectively and i, j, and k are the unit vectors along the coordinate axes. If x, y, and z axes are replaced respectively by x_1, x_2, and x_3 and the unit vectors along these directions are denoted by $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$, then the vector $\overline{V}$ may be represented as

$$\overline{V} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3$$

where v_1, v_2, and v_3 are the components of vector along the three directions. Using summation notation the above vector can be represented as

$$\overline{V} = \sum_{i=1}^{3} v_i\hat{e}_i = v_i\hat{e}_i, \qquad i = 1,2,3$$

where the repeating index i is called *dummy index*. Repetition of an index in a term implies a summation with respect to that index over its range. This short hand notation for summation is called *Einstein's summation convention*.

The i^{th} component of $\overline{V}$ can be written as v_i. If nonrepeating index appears in a term it is called a *free index* and summation is not implied in this case.

Kronecker delta

Kronecker delta, also known as *identity tensor* is defined as

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Also, since the unit vectors are linearly independent,

$$\hat{e}_i \cdot \hat{e}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Thus, we have

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

The most common use of the Kronecker delta is in the following operation: If we have terms in which one of the indices of δ_{ij} is repeated, then it simply replaces the dummy index by the other index of δ_{ij}. Consider

$$\delta_{ij} u_j = \delta_{i1} u_1 + \delta_{i2} u_2 + \delta_{i3} u_3$$

The right-hand side is u_1 when $i = 1$, u_2 when $i = 2$, u_3 when $i = 3$. Therefore

$$\delta_{ij} u_j = u_i$$

Dot product between two vectors

Let $\overline{V}$ and $\overline{W}$ are three-dimensional vectors, the dot product (inner product) between these vector is

$$\begin{aligned} \overline{V} \cdot \overline{W} &= (v_i \hat{e}_i) \cdot (w_j \hat{e}_j) \\ &= v_i w_j \delta_{ij} \qquad\qquad i, j = 1, 2, 3 \\ &= v_i w_i \\ &= v_1 w_1 + v_2 w_2 + v_3 w_3 \end{aligned}$$

Levy–Civita operator (permutation epsilon)

Levy–Civita operator, also known as *permutation epsilon* or *alternating unit tensor* is defined to be 1, 0, or -1, according to

$$\varepsilon_{ijk} := \begin{cases} 1 & \text{if } ijk \text{ are distinct and in cyclic order} \\ -1 & \text{if } ijk \text{ are distinct but not in cyclic order} \\ 0 & \text{if } ijk \text{ are not distinct (repeats)} \end{cases}$$

$$\hat{e}_i \times \hat{e}_j = \varepsilon_{ijk} \hat{e}_k$$

It follows directly from the definition of the Kronecker delta and the Levy–Civita operator, that the equation

$$\varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$$

is valid.

Cross product between two vectors

The cross product (vector product) between two vector is defined as

$$\begin{aligned} \overline{V} \times \overline{W} &= \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= (v_2 w_3 - v_3 w_2) \hat{e}_1 + (v_3 w_1 - v_1 w_3) \hat{e}_2 + (v_1 w_2 - v_2 w_1) \hat{e}_3 \end{aligned}$$

Using indicial notation,

$$\begin{aligned}\overline{V}\times\overline{W} &= (v_j\hat{e}_j)\times(w_k\hat{e}_k)\\ &= v_jw_k\,\hat{e}_j\times\hat{e}_k\\ &= v_jw_k\,\varepsilon_{ijk}\,\hat{e}_i\end{aligned}$$

The i^{th} component of $\overline{V}\times\overline{W}$ is given by $v_jw_k\,\varepsilon_{ijk}$. if $i=1$ we have

$$v_2w_3\varepsilon_{123}+v_3w_2\varepsilon_{132} = v_2w_3 - v_3w_2$$

Scalar triple product

The scalar triple product between three vectors $\overline{U}$, $\overline{V}$, and $\overline{W}$ is defined as

$$\begin{aligned}\overline{U}\cdot(\overline{V}\times\overline{W}) = &\begin{vmatrix} u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}\\ &= u_1(v_2w_3-v_3w_2)+u_2(v_3w_1-v_1w_3)+u_3(v_1w_2-v_2w_1)\end{aligned}$$

Using indicial notation,

$$\begin{aligned}\overline{U}\cdot(\overline{V}\times\overline{W}) &= u_i(\overline{V}\times\overline{W})_i\\ &= u_iv_jw_k\,\varepsilon_{ijk}\end{aligned}$$

Vector differential operators

Gradient

The gradient operator is defined as

$$\begin{aligned}\nabla &\equiv \hat{e}_1\frac{\partial}{\partial x_1}+\hat{e}_2\frac{\partial}{\partial x_2}+\hat{e}_3\frac{\partial}{\partial x_3}\\ &= \hat{e}_i\frac{\partial}{\partial x_i}\end{aligned}$$

Thus, the gradient of scalar is given by

$$\begin{aligned}\nabla\phi &= \hat{e}_i\frac{\partial\phi}{\partial x_i}\\ &= \hat{e}_1\frac{\partial\phi}{\partial x_1}+\hat{e}_2\frac{\partial\phi}{\partial x_2}+\hat{e}_3\frac{\partial\phi}{\partial x_3}\end{aligned}$$

The i^{th} component of $\nabla\phi$ is given by $\dfrac{\partial\phi}{\partial x_i}$.

Divergence

The divergence of a vector $\overline{V}$ is defined as

$$\begin{aligned}\nabla\cdot\overline{V} &\equiv \left(\hat{e}_i\frac{\partial}{\partial x_i}\right)\cdot(v_j\hat{e}_j) \;=\; \frac{\partial v_j}{\partial x_i}\delta_{ij}\\ &= \frac{\partial v_i}{\partial x_i}\\ &= \frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\end{aligned}$$

Curl

The curl of a vector $\overline{V}$ is defined as

$$\begin{aligned}\nabla\times\overline{V} &\equiv \left(\hat{e}_j\frac{\partial}{\partial x_j}\right)\times(v_k\hat{e}_k)\\ &= \hat{e}_i\frac{\partial v_k}{\partial x_j}\varepsilon_{ijk}\end{aligned}$$

The i^{th} component of $\nabla\times\overline{V}$ is given by $\dfrac{\partial v_k}{\partial x_j}\varepsilon_{ijk}$. If $i=1$, we have

$$(\nabla\times\overline{V})_1 \;=\; \frac{\partial v_3}{\partial x_2}\varepsilon_{123}+\frac{\partial v_2}{\partial x_3}\varepsilon_{132} \;=\; \frac{\partial v_3}{\partial x_2}-\frac{\partial v_2}{\partial x_3}$$

Laplacian operator

The Laplacian operator is defined as

$$\begin{aligned}\nabla^2 \equiv \nabla\cdot\nabla &= \left(\hat{e}_i\frac{\partial}{\partial x_i}\right)\cdot\left(\hat{e}_j\frac{\partial}{\partial x_j}\right) \;=\; \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}\right)\delta_{ij} \;=\; \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_i}\right)\\ &= \frac{\partial^2}{\partial x_i^2}\end{aligned}$$

Thus, the Laplacian of a scalar ϕ is defined as

$$\nabla^2\phi \;=\; \frac{\partial^2\phi}{\partial x_i^2} \;=\; \frac{\partial^2\phi}{\partial x_1^2}+\frac{\partial^2\phi}{\partial x_2^2}+\frac{\partial^2\phi}{\partial x_3^2}$$

and the Laplacian of a vector $\overline{V}$ is defined as

$$\begin{aligned}\nabla^2\overline{V} &= \frac{\partial^2\overline{V}}{\partial x_j^2} \;=\; \frac{\partial^2(v_i\hat{e}_i)}{\partial x_j^2}\\ &= \hat{e}_i\frac{\partial^2 v_i}{\partial x_j^2} \qquad \rightarrow \quad \text{(in Cartesian coordinates)}\\ &= \hat{e}_1\frac{\partial^2 v_1}{\partial x_j^2}+\hat{e}_2\frac{\partial^2 v_2}{\partial x_j^2}+\hat{e}_3\frac{\partial^2 v_3}{\partial x_j^2}\end{aligned}$$

Substantial derivative

The Substantial derivative operator D/Dt is defined as

$$\begin{aligned}\frac{D}{Dt} &\equiv \frac{\partial}{\partial t} + \overline{V}\cdot\nabla = \frac{\partial}{\partial t} + (v_i\hat{e}_i)\cdot\left(\hat{e}_j\frac{\partial}{\partial x_j}\right)\\ &= \frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x_j}\delta_{ij}\\ &= \frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x_i}\end{aligned}$$

Thus, the substantial derivative of a scalar variable is defined as

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + v_i\frac{\partial\phi}{\partial x_i}$$

and the substantial derivative of a vector field is defined as

$$\begin{aligned}\frac{D\overline{V}}{Dt} &= \frac{\partial\overline{V}}{\partial t} + v_j\frac{\partial\overline{V}}{\partial x_j}\\ &= \hat{e}_i\left[\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right] \qquad \rightarrow \quad \text{(in Cartesian coordinates)}\end{aligned}$$

The i^{th} component of $\dfrac{D\overline{V}}{Dt}$ is given by $\dfrac{\partial v_i}{\partial t} + v_j\dfrac{\partial v_i}{\partial x_j}$.

Second-order tensors

We know that a scalar can be represented by a single number, and a Cartesian vector can be represented by three numbers. However, there are other quantities, that need more than three components for a complete description. For example, the stress at a point in a material needs nine components for a complete specification because *two* directions are involved in its description. One of the directions specifies the orientation of the *surface* on which the stress is acting, and the other specifies the direction of the *force* on that surface. Because two directions are involved, two indices are required to represent a second-order tensor. For example, if τ_{ij} represents the $(ij)^{\text{th}}$ component of stress tensor, the first index i denotes the direction of the unit normal to the surface, and the second index j denotes the direction in which the force is being projected.

A second-order tensor $\overline{\overline{T}}$ using indicial notation can be written as

$$\begin{aligned}\overline{\overline{T}} &= \tau_{ij}\hat{e}_i\hat{e}_j\\ &= (\tau_{1j}\hat{e}_1 + \tau_{2j}\hat{e}_2 + \tau_{3j}\hat{e}_3)\hat{e}_j\\ &= (\tau_{11}\hat{e}_1 + \tau_{21}\hat{e}_2 + \tau_{31}\hat{e}_3)\hat{e}_1 + (\tau_{12}\hat{e}_1 + \tau_{22}\hat{e}_2 + \tau_{32}\hat{e}_3)\hat{e}_2 + (\tau_{13}\hat{e}_1 + \tau_{23}\hat{e}_2 + \tau_{33}\hat{e}_3)\hat{e}_3\end{aligned}$$

In matrix form the second-order tensor is

$$\overline{\overline{T}} = \begin{bmatrix}\tau_{11} & \tau_{12} & \tau_{13}\\ \tau_{21} & \tau_{22} & \tau_{23}\\ \tau_{31} & \tau_{32} & \tau_{33}\end{bmatrix}$$

The $(ij)^{\text{th}}$ component of tensor $\overline{\overline{T}}$ is given by τ_{ij}.

Tensor properties

- $\overline{\overline{S}}+\overline{\overline{T}} = (s_{ij}+\tau_{ij})\,\hat{e}_i\hat{e}_j$ i.e., the $(ij)^{\text{th}}$ component of $\overline{\overline{S}}+\overline{\overline{T}}$ is $s_{ij}+\tau_{ij}$
- $\overline{\overline{S}}+\overline{\overline{T}} = \overline{\overline{T}}+\overline{\overline{S}}$ i.e., $s_{ij}+\tau_{ij} = \tau_{ij}+s_{ij}$
- $a\overline{\overline{T}} = a\tau_{ij}\,\hat{e}_i\hat{e}_j$ i.e., the $(ij)^{\text{th}}$ component of $a\overline{\overline{T}}$ is $a\tau_{ij}$

Symmetric and skew-symmetric tensors

A tensor $\overline{\overline{T}}$ is called *symmetric* in the indices i and j if the components do not change when i and j are interchanged, that is, if $\tau_{ij} = \tau_{ji}$. The matrix corresponding to this second-order tensor is therefore symmetric about the diagonal and made up of only *six* distinct components. A tensor is called *skew-symmetric* if $\tau_{ij} = -\tau_{ji}$. Note that a skew-symmetric tensor must have zero diagonal terms, and off-diagonal terms must be mirror images. It is therefore made up of only *three* distinct components. Any tensor can be represented as the sum of a symmetric part and a skew-symmetric part; for example,

$$\tau_{ij} = \tfrac{1}{2}(\tau_{ij}+\tau_{ji}) + \tfrac{1}{2}(\tau_{ij}-\tau_{ji})$$

The operation of interchanging i and j does not change the first term, but changes the sign of the second term. Therefore the first term is called the symmetric part of $\overline{\overline{T}}$ and second term is called skew-symmetric part of $\overline{\overline{T}}$.

Vector as a tensor

There is a very important relation between a vector in three dimensions and the skew-symmetric second order tensor. It is easy to show that every vector can be associated with a skew-symmetric tensor, and vise-versa. For example, we can associate the vector

$$\overline{V} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

with an skew-symmetric tensor defined by

$$\overline{\overline{R}} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

where the two are related as

$$\begin{aligned} R_{ij} &= -\varepsilon_{ijk}v_k \\ v_k &= -\frac{1}{2}\varepsilon_{ijk}R_{ij} \end{aligned}$$

Dyadic product

If $\overline{V}$ and $\overline{W}$ are two vectors, the *dyadic product* of these vectors is second order tensor, in which the elements of the array are products of the components of the vectors. The dyadic product is represented by $\overline{V}\,\overline{W}$.

$$\begin{aligned}
\overline{V}\,\overline{W} &= v_i\,\hat{e}_i w_j\,\hat{e}_j \;=\; v_i w_j\,\hat{e}_i\hat{e}_j \\
&= \left(v_1 w_j \hat{e}_1 + v_2 w_j \hat{e}_2 + v_3 w_j \hat{e}_3\right)\hat{e}_j \\
&= (v_1 w_1 \hat{e}_1 + v_2 w_1 \hat{e}_2 + v_3 w_1 \hat{e}_3)\,\hat{e}_1 + (v_1 w_2 \hat{e}_1 + v_2 w_2 \hat{e}_2 + v_3 w_2 \hat{e}_3)\,\hat{e}_2 \\
&\quad + (v_1 w_3 \hat{e}_1 + v_2 w_3 \hat{e}_2 + v_3 w_3 \hat{e}_3)\,\hat{e}_3 \\
&= \begin{bmatrix} v_1 w_1 & v_1 w_2 & v_1 w_3 \\ v_2 w_1 & v_2 w_2 & v_2 w_3 \\ v_3 w_1 & v_3 w_2 & v_3 w_3 \end{bmatrix}
\end{aligned}$$

The $(ij)^{\text{th}}$ component of the dyadic product $\overline{V}\,\overline{W}$ is given by $v_i w_j$. The order of the two vectors that constitute the dyadic product is important. In general, $\overline{V}\,\overline{W} \neq \overline{W}\,\overline{V}$.

Gradient of a vector

With the concept of the dyadic product between two vectors which produces a tensor a new calculus operation can be performed on a vector to obtain a second order tensor. This new operation is the gradient of a vector, resulting a second-order tensor.

$$\begin{aligned}
\nabla\overline{V} &= \hat{e}_j \frac{\partial \overline{V}}{\partial x_j} = \hat{e}_j \frac{\partial(\hat{e}_i v_i)}{\partial x_j} = \hat{e}_i\hat{e}_j \frac{\partial v_i}{\partial x_j} \\
&= \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}
\end{aligned}$$

The $(ij)^{\text{th}}$ component of the tensor $\nabla\overline{V}$ is given by $\dfrac{\partial v_i}{\partial x_j}$.

Unit tensor

The base vectors $\hat{e}_i$, $\hat{e}_j$, and $\hat{e}_k$ can be represented as

$$\hat{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \hat{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \hat{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and the unit tensor $\hat{\delta}$ as

$$\hat{\delta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The $(ij)^{\text{th}}$ component of the unit tensor $\hat{\delta}$ is given by Kronecker delta, δ_{ij}. It may be noted that the unit tensor δ_{ij} is an *isotropic tensor* in the sense that its components are unchanged by a rotation of the frame of reference. (There is no isotropic tensor of first order. δ_{ij} is the only isotropic tensor of second-order. There is also only one isotropic tensor of third order that is the *alternating tensor* or *permutation epsilon*.)

Dyadic product between two unit tensors

The product of two unit vectors is referred to as a *unit dyad*.

$$\hat{e}_i\hat{e}_j = \delta_{im}\delta_{jn} = \delta_{ij}$$

For example,

$$\hat{e}_1\hat{e}_2 = \delta_{1m}\delta_{2n} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that the unit dyads $\hat{e}_i\hat{e}_j$ and $\hat{e}_j\hat{e}_i$ are different. There are nine quantities $\hat{e}_i\hat{e}_j$ and each of these is multiplied by the corresponding component of the dyadic product. It should also be noted that the $(ij)^{\text{th}}$ component is in general different from the $(ji)^{\text{th}}$ component. The unit dyads play the same role in tensor analysis that the unit vectors played in vector analysis. The notion of dyads can be generalized to tensors because all that matters is the entity which has nine unit dyads multiplied by the corresponding components.

Inner product or double dot product

The double dot product between two tensors is a scalar and is defined as

$$\begin{aligned} \overline{\overline{S}} : \overline{\overline{T}} &= (s_{ij}\,\hat{e}_i\hat{e}_j) : (\tau_{pq}\,\hat{e}_p\hat{e}_q) \\ &= s_{ij}\tau_{pq}(\hat{e}_i \cdot \hat{e}_p)(\hat{e}_j \cdot \hat{e}_q) \\ &= s_{ij}\tau_{pq}\,\delta_{ip}\delta_{jq} \\ &= s_{ij}\tau_{ij} \end{aligned}$$

The term $s_{ij}\tau_{ij}$ can be expanded as follows

$$\begin{aligned} s_{ij}\tau_{ij} &= s_{1j}\tau_{1j} + s_{2j}\tau_{2j} + s_{3j}\tau_{3j} \\ &= s_{11}\tau_{11} + s_{12}\tau_{12} + s_{13}\tau_{13} + s_{21}\tau_{21} + s_{22}\tau_{22} + s_{23}\tau_{23} + s_{31}\tau_{31} + s_{32}\tau_{32} + s_{33}\tau_{33} \end{aligned}$$

Thus, the double dot product between two tensors is obtained when their individual entries are multiplied with each other and summed. Note that another product $s_{ij}\tau_{ji}$ is also possible and is denoted by $\overline{\overline{S}} : \overline{\overline{T}}^T$. If $\overline{\overline{T}}$ is a symmetric tensor, then $s_{ij}\tau_{ij} = s_{ij}\tau_{ji}$ and hence

$$s_{ij}\tau_{ij} \;=\; s_{11}\tau_{11} + s_{22}\tau_{22} + s_{33}\tau_{33} + 2\,(s_{12}\tau_{12} + s_{13}\tau_{13} + s_{23}\tau_{23})$$

The inner product $s_{ij}\tau_{ij} = 0$ if one of these tensors is symmetric and the other is skew-symmetric. We also have,

$$\begin{aligned}
\overline{\overline{T}} : \overline{V}\,\overline{W} &= \tau_{ij} v_i w_j \\
\overline{V}\,\overline{W} : \overline{X}\,\overline{Y} &= v_i w_j x_i y_j \\
\overline{\overline{T}} : \hat{\delta} &= \tau_{ij}\delta_{ij} = \tau_{ii} \\
\overline{\overline{T}} : \nabla\overline{V} &= \tau_{ij}\frac{\partial v_i}{\partial x_j}
\end{aligned}$$

Dual vector of a tensor

The *dual vector* d_i of a tensor τ_{jk} is defined by the inner product

$$d_i = \varepsilon_{ijk}\tau_{jk}$$

It may be proved that this product is vector.

Dot product of tensor and vector

The dot product between a tensor and a vector (*contracted product*) is a vector.

$$\begin{aligned}
\overline{\overline{T}} \cdot \overline{V} &= (\tau_{ij}\hat{e}_i\hat{e}_j) \cdot (v_k\hat{e}_k) \\
&= \tau_{ij} v_k \hat{e}_i \delta_{jk} \\
&= \tau_{ij} v_j \hat{e}_i
\end{aligned}$$

The i^{th} component of $\overline{\overline{T}} \cdot \overline{V}$ is given by $\tau_{ij}v_j$.

Similarly, the j^{th} component of $\overline{V} \cdot \overline{\overline{T}}$ is given by $v_i\tau_{ij}$.

$$\overline{\overline{T}} \cdot \overline{V} = \overline{V} \cdot \overline{\overline{T}} \qquad \text{(for symmetric tensor)}$$

Tensor product of two tensors

The single dot product between two tensors is a tensor. For example,

$$\begin{aligned}
\overline{\overline{R}} &= \overline{\overline{S}} \cdot \overline{\overline{T}} \\
&= (s_{ip}\hat{e}_i\hat{e}_p) \cdot \left(\tau_{qj}\hat{e}_q\hat{e}_j\right) \\
&= s_{ip}\tau_{qj}\hat{e}_i\hat{e}_j\delta_{pq} \\
&= s_{ip}\tau_{pj}\hat{e}_i\hat{e}_j
\end{aligned}$$

where $\overline{\overline{R}}$ is a tensor. The $(ij)^{\text{th}}$ component of $\overline{\overline{R}}$ is, $r_{ij} = s_{ip}\tau_{pj}$.

Differential operations involving tensors

The calculus of tensors also follows the same lines as vector calculus. Consider some operations of tensor calculus. The divergence of a tensor is defined as the dot product of the gradient operator ∇ and the tensor. Consider evaluation in Cartesian coordinates:

$$\begin{aligned}\nabla \cdot \overline{\overline{T}} &= \left(\hat{e}_j \frac{\partial}{\partial x_j}\right) \cdot (\tau_{ki}\hat{e}_k\hat{e}_i) \\ &= \hat{e}_i \frac{\partial \tau_{ki}}{\partial x_j}\delta_{jk} \\ &= \hat{e}_i \frac{\partial \tau_{ji}}{\partial x_j}\end{aligned}$$

The i^{th} component of $\nabla \cdot \overline{\overline{T}}$ is given by $\frac{\partial \tau_{ji}}{\partial x_j}$. If τ_{ij} is symmetric,

$$\nabla \cdot \overline{\overline{T}} = \hat{e}_i \frac{\partial \tau_{ij}}{\partial x_j}$$

Note that while the divergence of a vector is a scalar the divergence of a tensor gives rise to a vector.

$$\begin{aligned}\overline{W} \cdot \nabla \overline{V} &= (w_j \hat{e}_j) \cdot \left(\hat{e}_k \frac{\partial (\hat{e}_i v_i)}{\partial x_k}\right) \\ &= \hat{e}_i w_j \frac{\partial v_i}{\partial x_k}\delta_{jk} \\ &= \hat{e}_i w_j \frac{\partial v_i}{\partial x_j}\end{aligned}$$

The i^{th} component of $\overline{W} \cdot \nabla \overline{V}$ is given by $w_j \frac{\partial v_i}{\partial x_j}$.

Another useful relation involving a tensor $\overline{\overline{T}}$ and vector $\overline{V}$ is as follows

$$\begin{aligned}\nabla \cdot \left(\overline{\overline{T}} \cdot \overline{V}\right) &= \overline{V} \cdot \left(\nabla \cdot \overline{\overline{T}}\right) + \overline{\overline{T}} : \nabla \overline{V} \\ \frac{\partial}{\partial x_i}(\tau_{ij} v_j) &= v_i \frac{\partial \tau_{ji}}{\partial x_j} + \tau_{ij} \frac{\partial v_i}{\partial x_j}\end{aligned}$$

For a symmetric tensor $\overline{\overline{T}}$, the above relation may be written as

$$\frac{\partial}{\partial x_i}(\tau_{ij} v_j) = v_i \frac{\partial \tau_{ij}}{\partial x_j} + \tau_{ij} \frac{\partial v_i}{\partial x_j}$$